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Nonlinear Processes in Geophysics

The modified Korteweg - de Vries equation in the theory of large - amplitude internal waves

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Abstract. The propagation of large- amplitude internal waves in the ocean is studied here for the case when the nonlinear effects are of cubic order, leading to the modified Korteweg - de Vries equation. The coefficients of this equation are calculated analytically for several models of the density stratification. It is shown that the coefficient of the cubic nonlinear term may have either sign (previously only cases of a negative cubic nonlinearity were known). Cubic nonlinear effects are more important for the high modes of the internal waves. The nonlinear evolution of long periodic (sine) waves is simulated for a three-layer model of the density stratification. The sign of the cubic nonlinear term influences the character of the solitary wave generation. It is shown that the solitary waves of both polarities can appear for either sign of the cubic nonlynear term; if it is positive the solitary waves have a zero pedestal, and if it is negative the solitary waves are generated on the crest and the trough of the long wave. The case of a localised impulsive initial disturbance is also simulated. Here, if the cubic nonlinear term is negative, there is no solitary wave generation at large times, but if it is positive solitary waves appear as the asymptotic solution of the nonlinear wave evolution.

1. Introduction

Large-amplitude internal waves are observed very often in different regions of the world oceans, mainly in coastal waters. Sometimes the solitary wave character of these waves can be determined, in particular, such a analysis was done for the Sulu sea (Apel et al, 1985), the North-West shelf of Australia (Holloway, 1987), the Okhotsk sea (Nagovitsyn et al, 1991), the Biscay Bay (New & Pingree, 1990), and the Canadian Shelf (Gan & Ingran, 1992; Sandstrom & Oakey, 1995). Interpretation of the observed data is usually produced from a weakly nonlinear wave theory, and the Korteweg - de Vries equation is the basic equation for the description of long nonlinear internal waves in the coastal zone. The coefficients of this equation are determined through the vertical structure of the density field and the shear flow in the ocean, which may be variable in space and time. Analysis of the variability of the coefficients of the Korteweg – de Vries equation for the eastern part of the Mediterranean (Pelinovsky et al, 1995), the Baltic Sea (Talipova et al, 1997a), the North-West shelf of Australia (Holloway et al, 1997) has shown that the coefficient of the quadratic nonlinear term can very significantly, and it may even change its sign in the coastal zone. The effect of the changing of the sign of the quadratic nonlinear term is well known for the idealised theoretical model of a two-layer fluid; in particular, it is negative if the pycnocline is close to the sea surface, and positive if the pycnocline is close to the seafloor. In general, the coefficient of the quadratic nonlinear term may vanish for certain critical combinations of the parameters of more complex models of the fluid density stratification. This changing of the sign of the quadratic nonlinear term leads to interesting features of the solitary wave transformation in such zones. In particular, to the destruction of the initial solitary wave of one polarity, and the generation of terminal solitary waves with the opposite

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polarity (Pelinovsky & Shavratsky, 1976; Djordjevic & Redekopp, 1978; Knickerbocker & Newell, 1980; Helfrich et al., 1985; Talipova et al, 1997b; Grimshaw et al, 1997). This is evidence that the role of the next order nonlinear terms in the evolution equation is increased in the zones with a weak quadratic nonlinear term, and both, cubic and quadratic nonlinearities maybe of comparable significance. Such corrections were obtained for a two-layer fluid (Djordjevic & Redekopp, 1978: Kakutani, Yamasaki, 1978; Miles, 1979, 1981; Koop & Butler, 1981) and the extended Korteweg - de Vries equation for the vertical displacement of the interface between layers with two different densities has the following form

$$\frac{\partial\eta}{\partial t}+(c+\alpha\eta+\alpha_1\eta^2)\frac{\partial\eta}{\partial x}+\beta\frac{\partial^3\eta}{\partial x^3}=0\,, \quad (1)$$

where its coefficients are (in the Boussinesq approximation)

$$c=\sqrt{\frac{g\Delta\rho}{\rho}\frac{h_1h_2}{h_1+h_2}}\,, \quad (2)$$

$$\beta=\frac{ch_1h_2}{6}\,, \quad (3)$$

$$\alpha=\frac{3c}{2}\frac{h_1-h_2}{h_1h_2}\,, \quad (4)$$

$$\alpha_1=-\frac{3c}{8h_1^2h_2^2}(h_1^2+h_2^2+6h_1h_2)\,. \quad (5)$$

Here $\Delta\rho/\rho$ is the density jump between the upper layer of thickness h_1 and the lower layer of thickness h_2 (total depth $H = h_1 + h_2$). As can be seen from (5), the coefficient of the cubic nonlinear term is always negative for either depth of the pycnocline location while the quadratic nonlinear term may be either, positive or negative depending on the pycnocline location. The extended Korteweg -- de Vries equation (1) with the negative cubic nonlinear term can be reduced to the "classic" Korteweg -- de Vries equation and solved by the inverse-scattering method (Miles, 1979, 1981). The main result here is the generation of solitary waves and oscillatory tails as in the "classic" Korteweg - de Vries equation, but the solitary waves have a limitation to their amplitude, and the solitary wave with an amplitude close to the limiting wave has a large width. Because a two-layer model of the density stratification is a very particular case, it is important to know the possible signs of the cubic nonlinear term in more general models of the ocean stratification. A general expression for the coefficient of the cubic nonlinear term for a fluid with arbitrary stratification of the density and shear flow was obtained by Lee & Beardsley, 1974 and Lamb & Yan, 1996, but it is very complex so that it is difficult to obtain a conclusion about possible signs of the cubic nonlinear term in the general case. Lamb & Yan (1996) considered one example of the density stratification with a pycnocline near to the sea surface, and this gives again a negative sign of the cubic nonlinear term. A more detailed analysis of the cubic nonlinear effects for a solitary wave is given by Gear & Grimshaw (1983). In particular, for the examples considered in this paper (for instance, flow with a constant shear, or with a linear buoyancy frequency), the second-order correction to the solitary wave speed is negative as for a two-layer fluid. But the solitary wave speed correction is a product of several second-order terms (cubic nonlinearity, nonlinear dispersion, and higher-order linear dispersion) and it is difficult to select in the numerical calculations the contribution of the cubic nonlinear term alone.

Here we will show that the cubic nonlinear term may, in general, have either sign (or can be equal to zero) depending on the density stratification. Several models of a fluid density stratification are considered in Section 3. The main feature of the density stratifications considered is their symmetry which leads to a zero value of the quadratic nonlinear term, and then the expressions for the cubic nonlinear term have a simple form. One of the models is a three-layer fluid with a symmetrical location of the identical density jumps from the middle layer. The cubic nonlinear term (as for all other coefficients of the modified Korteweg - de Vries equation) is calculated in explicit form for several models of the density stratification. It is shown that the cubic nonlinear term may have either sign, and that cubic nonlinear effects are more important for high modes of the internal waves. The steady-state solitary wave solutions of the modified Korteweg -- de Vries equation for different signs of the cubic nonlinearity are summarised in Section 4, and then used for the calculation of the nonlinear vertical structure of the internal solitary wave. Some results of numerical simulations of the evolution of a periodic wave are given in Section 5, and they show the influence of the sign of the cubic non-

Corrigenda

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In our paper a key part of formula (10) on page 239 was missing. The correct formula is,

$$\alpha_1 = -\frac{3}{2}\frac{\int (c-U)^2 W\,dz}{\int (c-U)\,(d\Phi/dz)^2\,dz},$$

$$W = 2\,(d\Phi/dz)^4 - 3\,(dT/dz)\,(d\Phi/dz)^2. \tag{10}$$

Here α_1 is the coefficient of the cubic term in the modified KdV equation (formula (8) in the published paper), that is

$$\frac{\partial \eta}{\partial t} + (c + \alpha_1\,\eta^2)\,\frac{\partial \eta}{\partial x} + \beta\,\frac{\partial^3 \eta}{\partial x^3} = 0.$$

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linearity on the solitary wave developing on an initial long wave. Results of the numerical simulation of the evolution of the initial localised disturbance are presented also.

2. Basic model

The nonlinear theory of internal waves in a stratified ocean, with an accuracy of up to the second order on the wave amplitude, was developed by Lee & Beardsley (1974), Gear & Grimshaw (1983), Vlasenko (1994) and Lamb & Yan (1996). For simplicity we will discuss here only the case, when the quadratic nonlinear term in the Korteweg - de Vries equation is equal to zero. This occurs for certain critical combinations of the parameters of the fluid density stratification, when (in the Boussinesq approximation)

$$\int_0^H (c-U)^2 (d\Phi/dz)^3 dz = 0. \quad (6)$$

Here $\Phi(z)$ is a modal function for the vertical displacement of fluid particle and c is the long wave phase speed which is an eigenvalue of the problem,

$$\frac{d}{dz}\left[(c-U)^2 \frac{d\Phi}{dz}\right] + N^2(z)\Phi = 0,$$

$$\Phi(0) = \Phi(H) = 0, \quad (7)$$

and $N(z)$ and $U(z)$ are the buoyancy frequency and the background shear flow respectively, H is the total depth, the axis z is directed upwards from the seafloor, and the modal function $\Phi(z)$ is normalised at its maximum value.

When the quadratic nonlinear term is zero, the evolution of the large - amplitude internal waves is described by the modified Korteweg - de Vries equation

$$\frac{\partial\eta}{\partial t} + (c + \alpha_1\eta^2)\frac{\partial\eta}{\partial x} + \beta\frac{\partial^3\eta}{\partial x^3} = 0. \quad (8)$$

Here η is the wave profile (on multiplying by the modal function $\Phi(z)$, it gives the leading order term for vertical displacement of the streamfunction), x is the horizontal coordinate, t is time, β is the dispersion parameter in Bousinesq approximation),

$$\beta = \frac{1}{2}\frac{\int (c-U)^2\Phi^2 dz}{\int (c-U)(d\Phi/dz)^2 dz}, \quad (9)$$

and α_1 is the coefficient of the cubic nonlinear term,

$$W = 2(d\Phi/dz)^4 - 3(dT/dz)(d\Phi/dz)^2,$$

where both integrals are from the seafloor to the sea surface. The function $T(z)$ determines the nonlinear correction to the vertical structure of the wave mode in the second approximation, and it is the solution of the ordinary differential equation

$$\frac{d}{dz}\left[(c-U)2\frac{dT}{dz}\right] + N2(z)T$$
$$= \frac{3}{2}\frac{d}{dz}\left[(c-U)^2\left(\frac{d\Phi}{dz}\right)^2\right] \quad (11)$$

with zero boundary conditions. The expression (10) for zero quadratic nonlinearity was produced by Gear & Grimshaw (1983), and it is follows also from Lamb & Yan (1996).

3. Illustrative examples

First of all, let us consider interfacial waves in a three-layer model with two symmetrical density jumps $\Delta\rho/\rho$ on each interface, the width of the upper and lower layers is h, and the total depth is H. There are no shear flows, and the configuration is shown in Fig. 1 (this example is described in the short note by Talipova et al, 1997c). For a such configuration two modes of the internal waves can be calculated, but only the first mode has a symmetry which provides a zero value for the quadratic nonlinear term for any relation between H and h, see (6). The vertical structure of the first mode can be found easily

$$\Phi(z) = z/h, \qquad (0 < z < h),$$

$$\Phi(z) = 1, \qquad (h < z < H-h),$$

$$\Phi(z) = \frac{H-z}{h}, \quad (H-h < z < H), \quad (12)$$

as well as the solution of the equation (11)

$$T(z) = -\frac{3z}{2h^2}\left(1 - \frac{H}{2h}\right), \quad (0 < z < h),$$

$$T(z) = -\frac{3}{2h^2}\left(z - \frac{H}{2}\right), \ (h < z < H - h),$$

$$T)z) = -\frac{3}{2h^2}\left(1 - \frac{H}{2h}\right)(z - H),$$
$$(H - h < z < H). \tag{13}$$

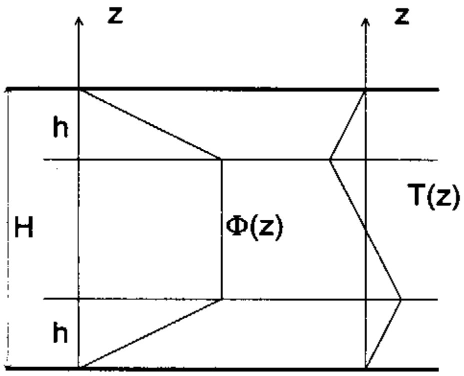


Fig. 1. The geometry problem.

Both functions are shown in Fig. 1 also. Using (12) and (13), the coefficients of the modified Korteweg - de Vries equation (8) are calculated as

$$c = \sqrt{\bar{g}h}, \quad \beta = \frac{ch}{4}\left(H - \frac{4h}{3}\right), \tag{14}$$

$$\alpha_1 = -\frac{3c}{4h^2}\left(13 - \frac{9H}{2h}\right). \tag{15}$$

where $\bar{g} = g\delta\rho / \rho$ is the reduced acceleration due to gravity.
Let us to analyse the expression (15) for the coefficient of the cubic nonlinear term. The function $\alpha_1(h/H)$ is presented in Fig. 2. When $h = H/2$ (a three-layer model transforms to a two-layer model) the expression (15) coincides with (5) as excepted, and gives the negative sign for the cubic nonlinear term. It is negative also if the thickness of the intermediate layer $d = H - 2h$ is less than the critical value

$$d_* = \frac{4}{13}H, \tag{16}$$

and it is positive if the width of the intermediate layer is more than this critical value. Therefore, the cubic nonlinear term can be both positive or negative for internal waves in a stratified fluid. The possibility of a positive sign of the cubic nonlinearity for internal waves seems not to have been previously noted. It is interesting to mention that for a fluid of large depth, both coefficients, dispersion and nonlinearity, are proportional to the total depth H, and therefore this scale can be eliminated. Thus, the nonlinear wave dynamics in deep water (but for the condition that the wavelength is more than total depth) does not depend on the total depth, which affects only the time-scale of the developing nonlinear processes.

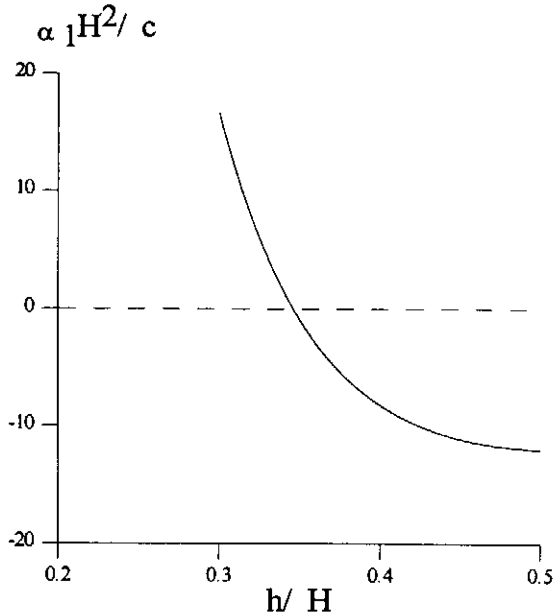


Fig. 2. The coefficient of the cubic nonlinear term for interfacial waves in a three-layer fluid.

The next example is again a three-layer fluid, where the upper and lower layers are stratified with a constant buoyancy frequency N and the intermediate layer is not stratified, while there is no density jump at each interface. We will consider again only the symmetrical modes for this model of the density stratification when the quadratic nonlinearity should be zero. The modal structure is found in the following form,

$$\Phi_n(z) = \sin\left(\frac{\pi}{2} + \pi n\right)\frac{z}{h}, \qquad (0 < z < h),$$

$$\Phi_n(z) = 1, \qquad (h < z < H - h),$$

$$\Phi_n(z) = \sin\left(\frac{\pi}{2} + \pi n\right)\frac{H - z}{h},$$

$$(H - h < z < H), \qquad (17)$$

where $2n + 1$ is the mode number. The nonlinear correction term is calculated from (11) in the lower layer

$$T(z) = \frac{1}{2h}\left(\frac{\pi}{2} + \pi n\right)\sin\left(1 + 2n\right)\frac{\pi z}{h} +$$

$$+(-1)^{n+1}\left(\frac{\pi}{2} + \pi n\right)^2 \frac{h - H/2}{h^2}\sin\left(\frac{\pi}{2} + \pi n\right)\frac{z}{h} \qquad (18)$$

in the middle layer

$$T(z) = \frac{1}{h^2}\left(\frac{\pi}{2} + \pi n\right)^2\left(\frac{H}{2} - z\right), \qquad (19)$$

and in the upper layer, txe same expression as in (18) holds, with z replaced by z-H.

The coefficients of the modified Korteweg - de Vries equation for this model of the density stratification are calculated in the explicit form

$$c_n = \frac{2Nh}{\pi + 2\pi n}, \qquad (20)$$

$$\frac{\beta_n}{c_n} = \frac{h(H - h)}{2(\pi/2 + \pi n)^2}, \qquad (21)$$

$$\frac{\alpha_1}{c_n} = \frac{6}{h^3}\left(\frac{\pi}{2} + \pi n\right)^2\left(\frac{H}{2} - h\right). \qquad (22)$$

It is interesting to point out here the coefficient of the cubic nonlinear term is positive for all values of the parameters, except in the trivial case $h = H/2$, when the buoyancy frequency is constant. In this degenerate case the nonlinearity is absent to all orders of the asymptotic theory (in the Boussinesq approximation). It is interesting to note that the dispersion coefficient is decreased for the high-order modes, meanwhile the nonlinear coefficient is increased. Therefore the role of the cubic nonlinearity for modes with the high numbers will be more significant. Also, here the ratio of the nonlinear coefficient to the dispersion coefficient tends to a constant value for deep water, and therefore, the nonlinear wave dynamics does not depend on the total depth.

The third example is a three-layer fluid as in the second example, but the buoyancy frequency in the middle layer has the value N_2, which differs from the buoyancy frequency N_1 in the upper and lower layers. For simplicity we will consider here only the first mode. The solution of the eigenvalue problem (7) can be found in terms of trigonometric functions, so that in the lower layer,

$$\Phi(z) = A\sin\left(\frac{N_1 z}{c}\right), \qquad (23a)$$

where

$$A = \frac{N_2}{N_1}\frac{\sin\left(\frac{N_2}{c}(H/2 - h)\right)}{\cos\left(\frac{N_1 h}{c}\right)}, \qquad (23b)$$

in the middle layer

$$\Phi(z) = \cos\left(\frac{N_2}{c}(z - H/2)\right), \qquad (24)$$

and in the upper layer, the same formula (23) with z replaced by H-z. This solution satisfies the normalisation condition $\Phi_{max} = 1$. The long wave phase speed is the solution of the algebraic transcendent equation,

$$\tan\frac{N_1 h}{c}\tan\left(\frac{N_2 h}{c}\left(\frac{H}{2h} - 1\right)\right) = \frac{N_2}{N_1}. \qquad (25)$$

Introducing new variables

$$\sigma = \frac{N_2}{N_1}, \quad \varphi = \frac{N_1 h}{c}, \quad \theta = \frac{H}{2h} - 1, \qquad (26)$$

the transcendental equation (25) can be rewritten as

$$\theta = \frac{1}{\sigma\varphi}a\tan\left(\frac{1}{\sigma\tan\varphi}\right), \qquad (27)$$

and defines the total depth as an explicit function of the long wave phase speed. This is very convenient for the analysis of these expressions. In particular, the limiting case $\varphi \to \pi/2$ corresponds to the one-layer fluid with the constant buoyancy frequency N_1, total depth is $H = 2h$ $(\theta = 0)$, while $c = N_1 H/\pi$. Another limit $\varphi \to 0$ corresponds to a fluid of large depth $(\theta \to \infty)$, when $c = N_2 H/\pi$ and there is no influence of the thin layers with the buoyancy frequency N_1. The function $c(H)$ obtained from (27) for different values of the parameter N_2/N_1 is shown in Fig. 3.

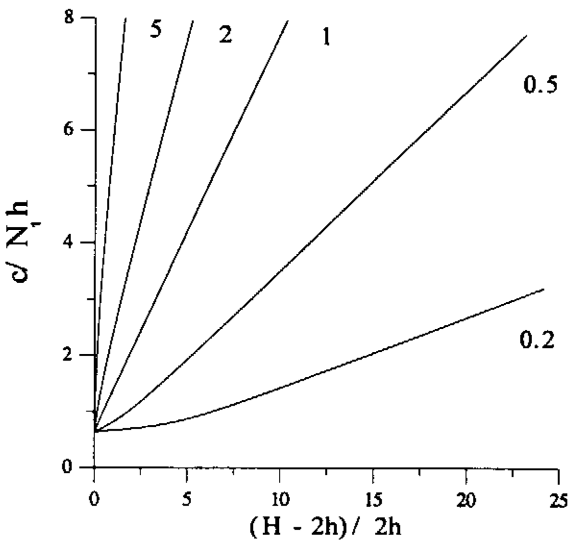


Fig. 3. The long wave phase speed for the three-layer stratification with different values of the buoyancy frequency. The numbers on the curves are the ratio N_2 / N_1.

The dispersion coefficient can be obtained as a function of the long wave phase speed,

$$\beta = \frac{ch^2}{2}\frac{I_2}{I_1}, \tag{28}$$

where

$$I_1 = \frac{\varphi}{2}\left\{\frac{\sigma^2 \sin^2(\sigma\theta\varphi)}{\cos^2\varphi}\left[\varphi + \frac{\sin(2\varphi)}{2}\right] + \right.$$

$$\left. + \sigma\left[\sigma\theta\varphi - \frac{\sin(2\sigma\theta\varphi)}{2}\right]\right\}, \tag{29}$$

$$I_2 = \frac{1}{2\varphi}\left\{\frac{\sigma^2 \sin^2(\sigma\theta\varphi)}{\cos^2\varphi}\left[\varphi - \frac{\sin(2\varphi)}{2}\right] + \right.$$

$$\left. + \frac{1}{\sigma}\left[\sigma\theta\varphi + \frac{\sin(2\sigma\theta\varphi)}{2}\right]\right\}. \tag{30}$$

In the limit case $\varphi \to \pi/2$ (a one-layer fluid with buoyancy frequency N_1) we have the asymptotic values: $I_1 \to (\pi^2/8)$, and $I_2 \to 1/2$, and in the other limiting case of deep water $(\varphi \to 0)$ we have $I_1 = (\pi\sigma\varphi/4)$, and $I_2 = (\pi/4\sigma\varphi)$. In both these limits, we have the expression for the dispersion coefficient

$$\beta = \frac{cH^2}{2\pi^2}, \tag{31}$$

which can be easily obtained from the definition for a fluid with a constant buoyancy frequency. The function $\beta(H)$ calculated from (27) and (28) is shown in Fig. 4 for different values of the parameter N_2/N_1.

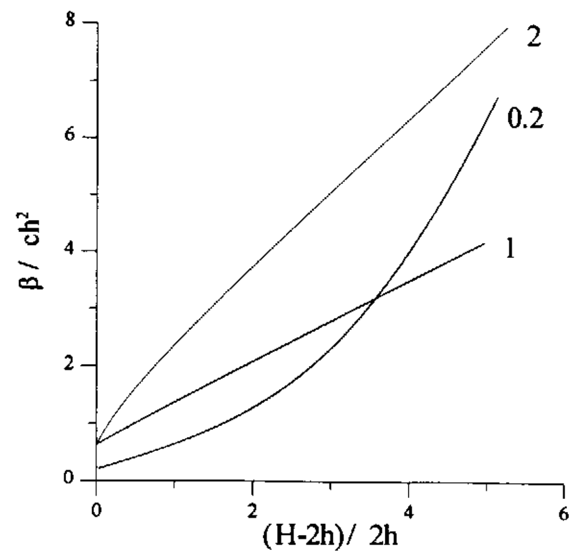


Fig. 4. The dispersion parameter for the three-layer stratification with different values of the buoyancy frequency. The numbers on the curves are the ratio N_2 / N_1.

The solution of equation (11) satisfying the condition $T(H/2) = 0$ is, in the lower layer,

$$T(z) = \frac{A^2 N_1}{2c}\sin\frac{2N_1 z}{c} + \frac{QN_1}{c}\sin\frac{N_1 z}{c}, \tag{32a}$$

and in the middle layer,

$$T(z) = \frac{N_2}{2c}\sin\left(\frac{2N_2}{c}\left(\frac{H}{2} - z\right)\right) -$$

$$-\frac{RN_2}{c}\sin\left(\frac{N_2}{c}\left(\frac{H}{2} - z\right)\right), \tag{32b}$$

where the constants Q and R are found from the condition of the continuity of T and dT/dz on the boundary $z = h$;

$$Q = \Delta_Q / \Delta, \quad R = \Delta_R / \Delta, \tag{33a}$$

and

$$\Delta = -\sigma^2 \sin\varphi \cos(\sigma\theta\varphi) - -\sigma\cos\varphi \sin(\sigma\theta\varphi), \tag{33b}$$

$$\Delta_Q = -\sigma^2 P_1 \cos(\sigma\theta\varphi) - \sigma P_2 \sin(\sigma\theta\varphi), \tag{33c}$$

$$\Delta_R = P_2 \sin\varphi - P_1 \cos\varphi, \tag{33d}$$

$$P_1 = \frac{\sigma}{2}\sin(2\sigma\theta\varphi) - \frac{A^2}{2}\sin(2\varphi), \tag{33e}$$

$$P_2 = -\sigma^2 \cos(2\sigma\theta\varphi) - A^2 \cos(2\varphi). \tag{33f}$$

As a result, the nonlinear coefficient in (10) can be expressed as

$$\alpha_{1_3} = -\frac{3c}{2h}\frac{2I_3 - 3I_4}{I_1}, \tag{34}$$

where

$$I = \frac{\varphi^3}{4}\left\{A^4\left[\frac{3\varphi}{2} - 2\sin(2\varphi) + \frac{1}{8}\sin(4\varphi)\right] + \right.$$

$$\left. +\sigma^3\left[\frac{3}{2}\sigma\theta\varphi - \sin(2\sigma\theta\varphi) + \frac{1}{8}\sin(4\sigma\theta\varphi)\right]\right\}, \tag{35}$$

and

$$I_4 = A^2\varphi^3\left\{\frac{A^2}{4}\left[\varphi + \sin(2\varphi) + \frac{1}{4}\sin(4\varphi)\right] + \right.$$

$$\left. +Q\left[\sin\varphi - \frac{1}{3}\sin^3\varphi\right]\right\} + +\sigma^3\varphi^3\left[\frac{1}{4}\sigma\theta\varphi - \right.$$

$$-\frac{1}{4}\sin(2\sigma\theta\varphi) + \frac{1}{16}\sin(4\sigma\theta\varphi) +$$

$$\left. +\frac{R}{3}\sin^3(\sigma\theta\varphi)\right]. \tag{36}$$

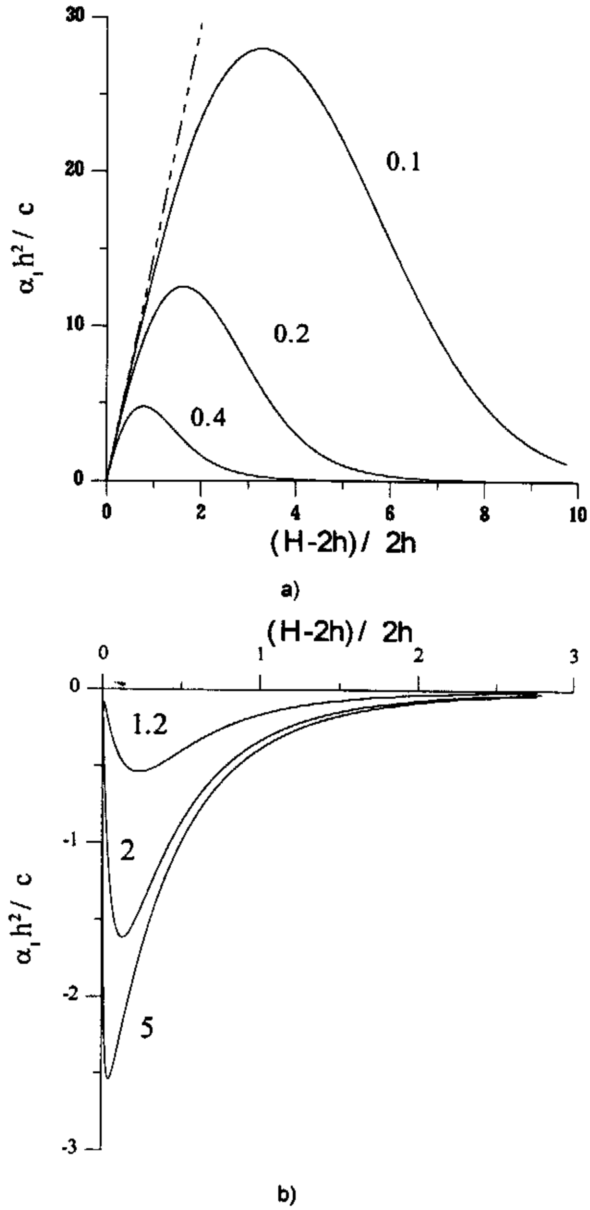


Fig. 5. The nonlinear parameter for the three-layer stratification with different values of the buoyancy frequency. The numbers on the curves are the ratio N_2/N_1.

The expression for the coefficient of the cubic nonlinear term is relatively complex. In both the limiting cases considered above ($H = 2h$, and $H \to \infty$), it is zero, because the density stratification tends to a single stratified layer with constant buoyancy frequency. In the general case we calculated it for different values of the parameter N_2/N_1, see Fig. 5. The nonlinear term is positive for the case $N_2 < N_1$ (Fig. 5a). If this ratio is very small, we have the second example of the density stratification, when two stratified layers are separated by one non-stratified layer. In this case according to (22), the nonlinear coefficient grows linearly with the depth, and this function is shown in Fig. 5a by the dashed line. Any (even weak) stratification of the middle layer leads to the existence of a maximum for the nonlinear coefficient at intermediate depths. If $N_2 > N_1$, the nonlinear coefficient is negative (Fig. 5b), and its extreme value moves to the y - axis when N_2 / N_1 increases. In the limit case $N_1 = 0$ the nonlinear coefficient decreases as the total depth increases and this case is similar to the "classic" two-layer model, see (5) at $h_1 = h_2$. Here we can see a large difference between the values of the modulus of the nonlinear coefficient in the negative and positive regions, where the positive values of the cubic nonlinear term can be much more than the modulus of the negative values of the nonlinear coefficient (compare the first example (15)).

4. Solitary waves

The modified Korteweg - de Vries equation is an integrable equation and its solutions can be obtained by the inverse scattering method (see, for instance, Ablowitz & Clarkson, 1991). Let us give here a brief summary of the solitary wave solutions which will be used later for the calculation of the full two - dimensional structure of solitary internal waves, and for the interpretation of our results from numerical simulations of the modified Korteweg - de Vries equation.

The solitary wave solutions for both signs of the cubic nonlinear term were obtained by Perelman et al (1974a) and Ono (1976a,b). They can be written in the general form,

$$\eta = \eta_0 + \frac{2(b^2 - \eta_0^2)}{\eta_0 \pm b\cosh[\gamma(x - Vt)]}, \qquad (37)$$

where

$$b = \sqrt{\eta_0^2 + \frac{3\beta\gamma^2}{2\alpha_1}}, \qquad (38)$$

and the speed is given by,

$$V = c + \alpha_1 \eta_0^2 + \beta\gamma^2. \qquad (39)$$

Here ± corresponds to the different branches of the solitary wave. This solution defines a two-parameter family of solitary waves which depend on the pedestal η_0 and on the characteristic wavenumber γ. We introduce the solitary wave amplitude A (relative to the pedestal) where

$$A = 2(\pm b - \eta_0). \qquad (40)$$

Let us now consider separately different signs of the cubic nonlinear coefficient a_1.

$\alpha_1 > 0$ (positive cubic nonlinearyty)

For this case, yf $\eta_0 > 0$, it is easy to show from (38) and (40) that the "+" branch defines a solitary wave of positive polarity, which exists for all $A > 0$, but the "-" branch defines a solitary wave of negative polarity only for $|A| > 4\eta_0$. This result can be explained by considering small-amplitude solitary waves on the pedestal. In this case the modified Korteweg - de Vries equation transforms to the "classic" Korteweg - de Vries equation with a positive coefficient of the quadratic term, and its solution is a solitary wave of positive polarity, but not negative polarity. In the limit case of $|A| = 4\eta_0$, the "-" branch transforms to the algebraic solitary wave,

$$\eta = \eta_0 \left(1 - \frac{4}{1 + \frac{2\alpha_1 \eta_0^2}{3\beta}(x - \alpha_1 \eta_0^2 t)^2} \right). \qquad (41)$$

This algebraic solitary wave is unstable (Pelinovsky & Grimshaw, 1997). If the pedestal is negative ($\eta_0 < 0$), the "-" branch exists for all $A < 0$, and the "+" branch only for amplitudes $A > 4|\eta_0|$.

If there is no pedestal (or the wave amplitude is very large), the solution (37) transforms into

$$\eta = \frac{A}{\cosh\left(\sqrt{\frac{\alpha_1 A^2}{6\beta}}(x - Vt)\right)},$$

$$V = c + \frac{\alpha_1 A^2}{6}, \qquad (42)$$

and exists for both branches. It is important to note the solitary wave may have either polarity for the positive cubic nonlinearity, but the nonlinear correction to the speed is positive for either polarity.

N-soliton solutions of the modified Korteweg - de Vries equation with a positive nonlinear coefficient were obtained by Hirota (1972). The solution of the Cauchy problem for a localised initial disturbance (on a zero pedestal) is described by Perelman et al (1974a). Any local initial disturbance will at large times transform into a set of separated solitary waves and a spreading oscillating tail.

$\alpha_1 < 0$ (negative cubic nonlinearity)
In this case there is no solitary wave solution of the modified Korteweg - de Vries equation if there is no pedestal ($\eta_0 = 0$). The steady-state solution in this case is the dissipationless shock wave (Perelman et al, 1974a,b; Ono, 1976a)

$$\eta = \pm A \tanh\left(\sqrt{\frac{|\alpha_1| A^2}{6\beta}}\left[x - (c - \frac{|\alpha_1| A^2}{3})t\right]\right), \tag{43}$$

with exists for both branches. The solitary wave can only exists on a nonzero pedestal, see (38). The solitary wave is a wave of elevation on a negative pedestal ("-" branch), and a wave of depression on a positive pedestal ("+" branch). In the case of small amplitude it is the "usual" solitary wave of the "classic" Korteweg - de Vries equation, where the quadratic nonlinear term is due to the pedestal. With increasyng amplitude the solitary wave reaches the limit of a very wide wave (the limiting wave is a superposition of two shock waves (43) of different signs) with an amplitude $2\eta_0$ above (below) the pedestal.
N-soliton solutions of the modified Korteweg - de Vries equation with a negative cubic nonlinearity are given by Perelman et al. (1974a,b) and Ono (1976a). The interesting phenomenon of the elastic interaction of the solitary wave (37) with the shock wave (43) is described in these papers: the solitary wave after an interaction changes its polarity. The numerical solution of a Cauchy problem for the local initial disturbance shows that a small disturbance transforms into a sequence of solitary waves and a spreading tail, while a large disturbance transforms into a single "limiting solitary wave" and a spreading tail (Perelman et al, 1974a). The same result within the inverse scattering method was obtained by Miles (1981).

Vertical structure
In order to use there results in practice, we first note that the modified Korteweg - de Vries equation (8) is written for the wave profile $\eta(x,t)$, which in the linear long wave limit is the amplitude of the streamfunction for a linear mode. Better, to second order in accuracy the vertical displacement of a streamline from a height z above the bottom is (Gear & Grimshaw, 1983)

$$\varsigma(z,x,t) = \eta(x,t)\Phi(z) + \eta^2(x,t)T(z)\,. \tag{44}$$

In our calculations we have chosen $\Phi(H/2) = 1$ and $T(H/2) = 0$ (the value of the cubic nonlinear term does not depend on this condition), and, therefore, $\eta(x,t)$ is the nonlinear wave disturbance of the streamfunction at mid-depth. For other depths the wave disturbance of the streamfunction will differ from the function $\eta(x,t)$, calculated from the modified Korteweg - de Vries equation. For instance, we will consider here the case of positive cubic nonlinearity and use the explicit formulas (12) and (13) for the modal structure and the function $T(z)$ for interfacial waves in the three-layer fluid. The solitary wave disturbances for three depths being the mid-depth and the upper and lower boundaries between the layers, are as follows,
$z = h$

$$\varsigma = \frac{A}{\cosh\left(\sqrt{\frac{\alpha_1 A^2}{6\beta}}(x - Vt)\right)} - \frac{3}{2h}\left(1 - \frac{H}{2h}\right)\frac{A^2}{\cosh^2\left(\sqrt{\frac{\alpha_1 A^2}{6\beta}}(x - Vt)\right)}, \tag{45}$$

$z = H/2$

$$\varsigma = \frac{A}{\cosh\left(\sqrt{\frac{\alpha_1 A^2}{6\beta}}(x - Vt)\right)}, \tag{46}$$

$z = H - h$

$$\varsigma = \frac{A}{\cosh\left(\sqrt{\frac{\alpha_1 A^2}{6\beta}}(x - Vt)\right)} + $$

$$+\frac{3(H-h)}{2h^2}(1-\frac{H}{2h})\frac{A^2}{\cosh^2\left(\sqrt{\frac{\alpha_1 A^2}{6\beta}}(x-Vt)\right)} \qquad (47)$$

As can be seen, the wave disturbances at different depths differ both in the wave form and in the amplitude. For a solitary wave of positive polarity the amplitude of the wave disturbance is maximal on the lower boundary, and for a solitary wave of negative polarity the wave disturbance is maximal for the upper boundary. As a result we can say that the wave energy in the nonlinear wave propagates not only in the horizontal direction, but may also oscillate in the vertical direction (this effect is absent in the linear theory).

The function $T(z)$ in this sense can be interpreted as a nonlinear correction to the vertical structure of the nonlinear waves. Because the function $T(z)$ can be presented as a series of eigenfunctions $\Phi_n(z)$, sometimes such a solitary wave is called a multimodal nonlinear wave (Vlasenko, 1994), but it is better to speak about the nonlinear correction to the modal structure. It is important to mention that the modified Korteweg - de Vries equation is symmetrical about the sign of a wave profile (it is invariant with replacing η on $-\eta$), but the wave field at different depths remembers the quadratic nonlinearity of the basic equations and it is not invariant to replacing of the wave sign.

5. Nonlinear evolution of the periodic and localised disturbances

The numerical simulation of the modified Korteweg - de Vries equation (8) was done for two initial disturbances. The first case corresponds to a localised sinusoidal disturbance (with a phase variation only from zero to 2π) of wave length 1 km in a basin of 100 m depth. The numerical domain for this case has length 5 km and periodic boundary conditions were used. In the second case boundary conditions correspond to a periodic sine wave with wavelength 10.8 km. For simplicity the case of a three-layer fluid with two symmetrical density jumps was considered; the density jump $\Delta\rho/\rho$ was 0.00023. The width of the middle (non-stratified) layer is changed so that we have different signs for the cubic nonlinearity.

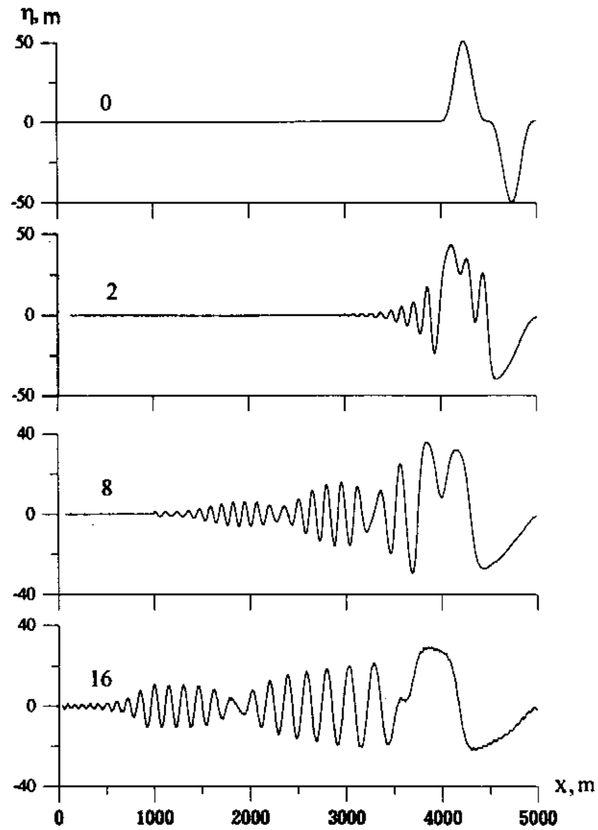


Fig. 6. Evolution of an initial localised disturbance for a negative sign of the coefficient of the cubic nonlinear term.

In the first experiment the distance between two layers is 22 m, and for this case the coefficients of the modified Korteweg - de Vries equation are as following, $\bar{n}$ = 0.3 m/sec, β = 140 m^3/sec, α = 0, α_1 = - 2×10^{-4} m^{-1}sec^{-1} . Here the modified Korteweg - de Vries equation has no solitary wave solution on a zero pedestal (see Section 4), and therefore a localised disturbance with zero asymptotic behaviour at both ends should decrease, transforming to a self-similar oscillatory tail. This dynamics is shown in Fig. 6 for different times (the time unit is 5 min). The initial amplitude of the impulse disturbance was chosen as 50 m comparable to the total depth so that the nonlinear effects can be manifested in a relatively small time. But the magnitude of the nonlinearity within the modified Korteweg - de Vries equation is determined by the parameter $|\alpha_1 A|/c$ which is 3.3×10^{-3} and, therefore, the nonlinearity is small. In the first stage due to the large initial amplitude, the process of generation of solitary waves of negative polarity on the

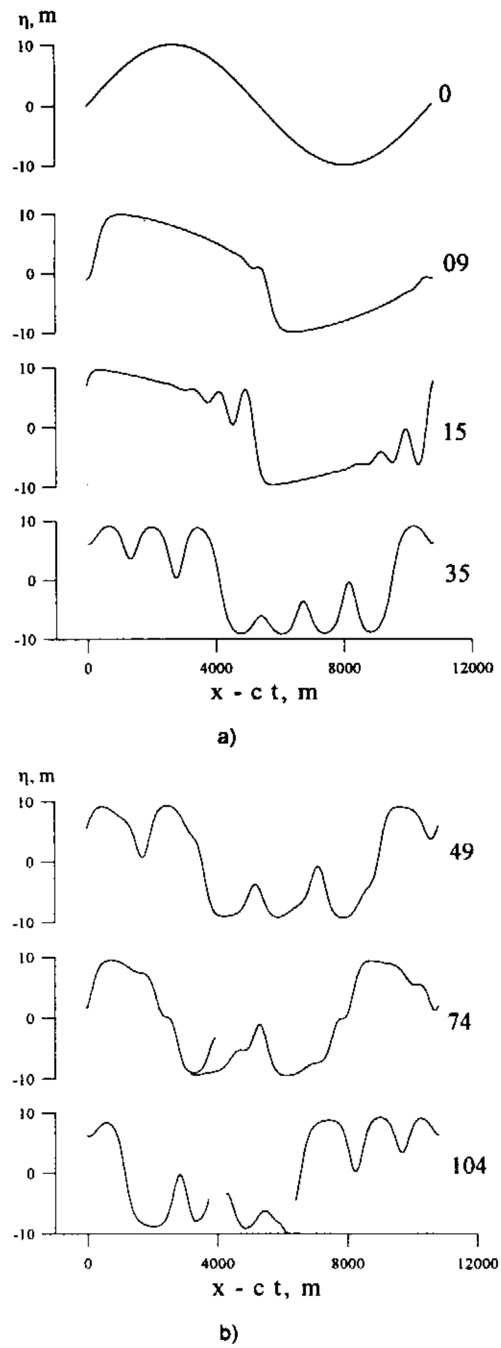


Fig. 7. Periodic wave transformation at the negative cubic nonlinearity.

crest of the positive part of the initial disturbance is visible. This process of the genertion of solitary waves on non-zero pedestals is very interesting and we specially consider the evolution of a periodic long wave. The wave forms are shown in Fig. 7 for different times
(the time unit here is 2.5 hr). The nonlinearity in first stage leads to the formation of two shocks on the long wave on the left slopes of each semi-wavelength, because the nonlinear correction to the wave speed is negative for disturbances of either sign, see (39). Then solitary waves of both polarities are generated on both shocks and situated on the crest and trough of the long wave. As was described in Section 4, solitary waves with a zero pedestal are impossible for this modified Korteweg - de Vries equation, but we have effectively variable pedestals related to the long wave and therefore a large effective quadratic nonlinearity. This explains the generation of the solitary waves of both polarities in the periodic wave field. Their amplitudes are less than the critical one (for the "limiting wave") and the picture is quite similar to those for the "classic" Korteweg - de Vries equation. Then, the solitary waves interact among themselves and with the shock waves, their number is decreased and they are rearranged (Fig. 7b).

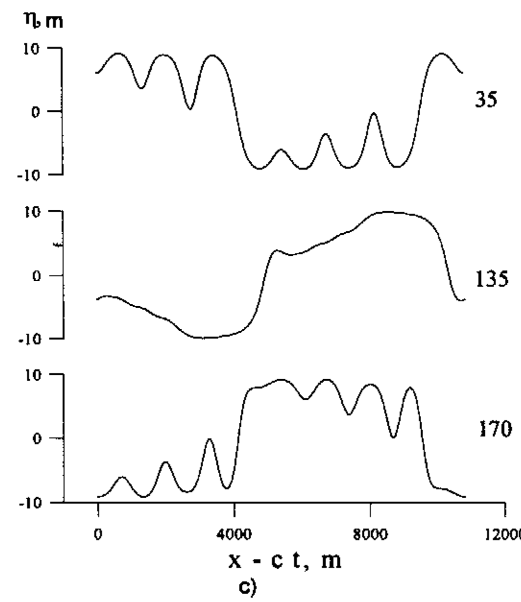


Fig. 7. Periodic wave transformation at the negative cubic nonlinearity (continued).

We see also the shifting of the dissipationless shocks to the left. After that, we can see the process of a near recurrence of the initial state and again, the generation of solitary waves (Fig. 7c). Thus, the generation of both dissipationless shocks and solitary waves, and their interaction is possible in a periodyc wave field for this case, when the cubic nonlinearity is negative.

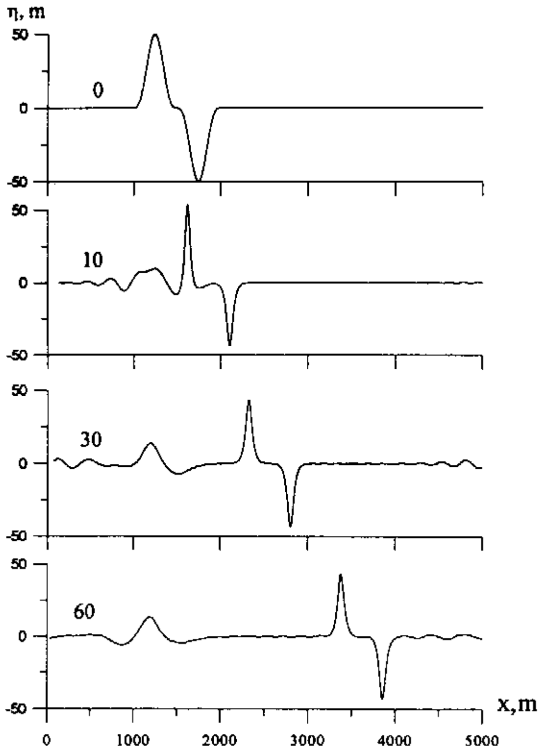


Fig. 8. Evolution of an initial localised disturbance for a positive sign of the coefficient of the cubic nonlinear term.

The thickness of the middle layer in the next experiment was 40 m, and this leads to a positive value of the cubic nonlinearity. Now, the coefficients of the modified Korteweg - de Vries equation (8) are as following, $\bar{n}$ = 0.26 m/sec, β = 120 m^3/sec, $\alpha = 0$, $\alpha_1 = 4\times10^{-4}$ $m^{-1}sec^{-1}$. The process of the evolution of a localised sign-variable disturbance is illustrated in Fig. 8 (the time unit is 5 min). The initial disturbance transforms in two large solitary waves of both polarityes and a small positive solitary wave. This last solitary wave has a small speed (it is proportional to A^2 for the solitary wave) and it is practically not shifted. Results of the same simulation for a periodic initial long wave is shown in Fig. 9 for different times (the time unit is 2.5 hr). Two shocks are formed again in an initial stage, but in the opposite phase of the long wave, because the nonlinear correction to the wave speed is now positive (Fig. 9a). Then solitary waves of the corresponding polarity are generated on each shock. Here all solitary waves have a zero pedestal, and are similar to (42). Large-amplitude solitary waves cross the small-amplitude solitary waves due to the difference in the wave speeds which do not depend on the polarity of the solitary wave. As a result, the solitary waves of opposite polarities are interleaved among themselves (Fig. 9a at t = 35). After that, we have a complex picture of the interaction of these solitary waves with a changing of their amplitudes (Fig. 9b), and a partial recurrence of the initial sine wave (Fig. 9c).

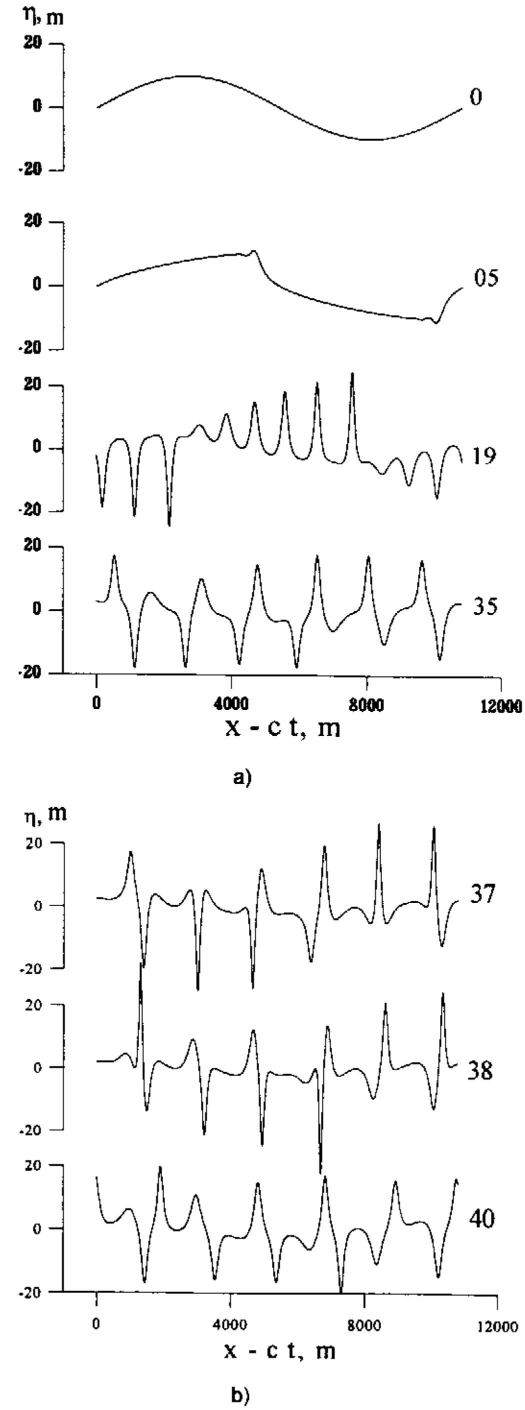
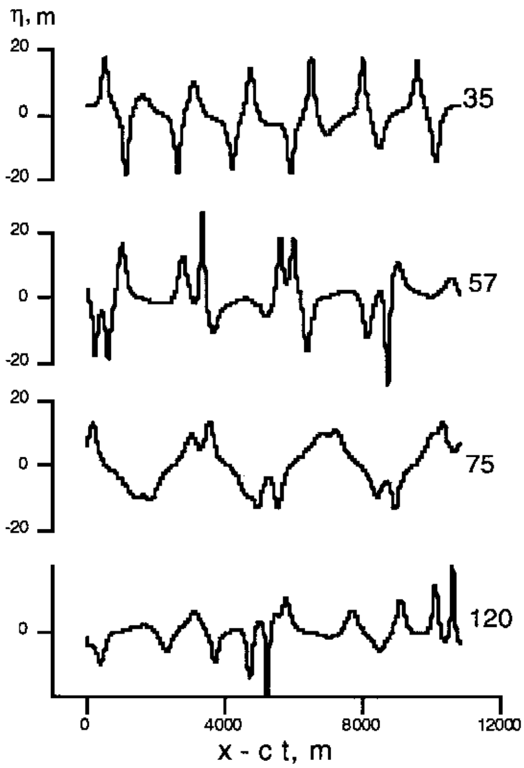


Fig. 9. Periodic wave transformation at the positive cubic nonlinearity.



c)

Fig. 9. Periodic wave transformation at the positive cubic nonlinearity (continued).

Numerical simulation confirms that the picture of solitary wave generation depends on the sign of the cubic nonlinear term quite significantly. It is important to note that the solitary waves in the periodic long wave field appear for either sign of the cubic nonlinear term, and they can have a non-zero pedestal.

6. Conclusion

We have considered models of the density stratification of a fluid for which the cubic nonlinearity can have uither sign; in particular, there are cases with a positive sign (previously only cases with a negative cubic nonlinearity were known). The sign of the cubic nonlinearity depends on the profile of the vertical density distribution. The cubic nonlinearity for internal waves of high modes is more significant. From our numerical simulation, we conclude that different signs of the cubic nonlinearity lead to different scenarios for the evolution of long waves. If the cubic nonlinearity is negative, solitary waves of both polarities and dissipationless shock waves are generated, and interact among themselves. If the cubic nonlinearity is positive, the solitary waves are generated on a zero pedestal, and also interact among themselves. Partial recurrence of the initial state is observed.

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