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Phase space structure and fractal trajectories in 1½ degree of freedom Hamiltonian systems whose time dependence is quasiperiodic

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Abstract. We consider particle motion in nonautonomous 1 degree of freedom Hamiltonian systems for which $H(p,q,t)$ depends on N periodic functions of t with incommensurable frequencies. It is shown that in near-integrable systems of this type, phase space is partitioned into nonintersecting regular and chaotic regions. In this respect there is no difference between the $N = 1$ (periodic time dependence) and the $N = 2, 3, \ldots$ (quasiperiodic time dependence) problems. An important consequence of this phase space structure is that the mechanism that leads to fractal properties of chaotic trajectories in systems with $N = 1$ also applies to the larger class of problems treated here. Implications of the results presented to studies of ray dynamics in two-dimensional waveguides and particle motion in two-dimensional incompressible fluid flows are discussed.

1 Introduction

Nonautonomous one degree of freedom Hamiltonian systems – often referred to as $1\frac{1}{2}$ degree of freedom systems – arise naturally in the description of at least two important geophysical systems. These are studies of ray dynamics in two dimensional waveguides (Abdullaev and Zaslavsky, 1991, 1993; Brown et al., 1991; Keers et al., 1997; Smith et al., 1992) and studies of particle motion in two dimensional incompressible flows (Aref, 1984; Brown and Smith, 1991; Brown and Samelson, 1994; del-Castillo-Negrete and Morrison, 1993; Osborne et al., 1986; Ottino, 1990; Pierrehumbert, 1991; Ridderinkhof and Zimmerman, 1992). Most theoretical studies of systems of this type assume that the dependence of the environment (via the Hamiltonian function) on the time-like variable is periodic. This assumption allows well known results such as the KAM theorem (Arnold, 1989; Tabor , 1989) to be exploited to provide insight into the underlying dynamics. Unfortunately, the geophysical systems that motivate these studies generally have structure which is not consistent with the assumption of periodicity. In this study we consider a much less restrictive class of $1\frac{1}{2}$ degree of freedom Hamiltonian systems – those for which the Hamiltonian is a function of N periodic functions of the time-like variable with incommensurable frequencies. Systems of this type – hereafter referred to as quasiperiodic – can be used to realistically describe commonly encountered environments in both types of geophysical problem mentioned above. It is shown that phase space in quasiperiodic systems has the same qualitative features as phase space in the more restrictive (periodic) class that has been extensively studied. One important consequence of this phase space structure is that the mechanism that leads to fractal properies of chaotic trajectories in periodic systems (Shlesinger et al., 1993; Zaslavsky et al., 1997) also applies to quasiperiodic systems. Some geophysical implications of this result will be discussed.

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Before proceeding, it is useful to provide some quantitative background material. We are concerned with one degree of freedom Hamiltonian systems,

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \tag{1}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{2}$$

where it is assumed that H depends explicitly on time t, $H = H(p,q,t)$. Recall that it follows from application of the chain rule and Eqs. 1 and 2 that the rate of change of H following a trajectory is equal to the local time rate of change of H,

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}. \tag{3}$$

It is well known (see, e.g., Lichtenberg and Lieberman, 1983) that this nonautonomous one degree of freedom system can be transformed to an autonomous two degree of freedom system,

$$\frac{dq_i}{d\tau} = \frac{\partial \bar{H}}{\partial p_i}, \qquad i = 1, 2 \tag{4}$$

$$\frac{dp_i}{d\tau} = -\frac{\partial \bar{H}}{\partial q_i}, \qquad i = 1, 2. \tag{5}$$

Here $p_1 = p$, $p_2 = -H$, $q_1 = q$, $q_2 = t$, τ is the new independent variable, and

$$\bar{H}((p_1, p_2), (q_1, q_2)) = H(p_1, q_1, q_2) + p_2. \tag{6}$$

The second of Eqs. 4 gives $dt/d\tau = 1$; with this condition the remaining equations reproduce Eqs. 1, 2 and 3.

The transformed system (4-6) has a bounded phase space only for the special case in which $H(p, q, t)$ is a periodic function of t; in this case t can be defined modulo one period, thereby setting bounds on q_2. Only for this special class of Hamiltonians $H(p, q, t)$ can the KAM theorem (which guarantees the existance of a dense set of nonchaotic trajectories for sufficiently small perturbation to a time-independent Hamiltonian) be applied to the transformed system (4-6). For this special class of Hamiltonians, Poincaré sections for the system (4-6) can be constructed by plotting the $(p_1, q_1) = (p, q)$ coordinates of one or more trajectories at integer multiples of the period of H. Because of the importance of working in a bounded phase space, the simplifying assumption that $H(p, q, t)$ is periodic in t is frequently introduced. Unfortunately, as noted above, the assumption that H is a periodic function of t often poorly approximates the geophysical systems that one would like to study.

In the next section it is shown that the periodicity assumption can be relaxed while maintaining a bounded phase space. An alternative transformed system is introduced to which the KAM theorem applies. Furthermore, it is shown that the partitioning of phase space into nonintersecting regular and chaotic regions that characterizes periodic systems carries over to the more general (quasiperiodic) class of problems treated here. Some implications of this phase space structure to the previously mentioned geophysical applications are discussed in section 3. Our results are summaruzed in the final section.

Previously, Beigie et al. (1991) have studied quasiperiodically forced dynamical systems. The focus of that work was a detailed local analysis of trajectories, including an analysis of the intersections of stable and unstable manifolds in chaotic regions and the corresponding lobe structure, and the application of Melnikov analysis. In contrast, the focus of our work is the qualitative structure of phase space (the coexistance of nonintersecting regular and chaotic regions) and the consequences of this structure on the long-time aysmptotic behavior of trajectories in geophysical systems.

2 Quasiperiodic Hamiltonian systems

In this section we consider systems of the form (1, 2) for which $H(p, q, t)$ depends on N periodic functions of t. N is assumed to be finite. The nonzero frequencies σ_i, $i = 1, 2, \ldots N$ may be (but need not be) incommensurable. (The special case for which the frequencies are commensurable can be reduced to the problem for which H is periodic in t, i.e., the $N = 1$ problem. Note, however, that if the period is long relative to all times of interest in the problem being studied, then this periodicity is of little practical value. Under such conditions, it is useful to apply the results described below. We shall focus on the incommensurable frequency problem because this is the most general problem to which our results apply.) Rather than transforming the system to (4-6), we transform the system to an autonomous $N+1$ degree of freedom system whose phase space is bounded:

$$\frac{dq_i}{d\tau} = \frac{\partial \bar{H}}{\partial p_i}, \qquad i = 1, 2, \ldots N+1, \tag{7}$$

$$\frac{dp_i}{d\tau} = -\frac{\partial \bar{H}}{\partial q_i}, \qquad i = 1, 2, \ldots N+1. \tag{8}$$

Here $p_i = -H/\sigma_i$, $i = 1, 2, \ldots N$, $p_{N+1} = p$, $q_i = \sigma_i t$, $i = 1, 2, \ldots N$, $q_{N+1} = q$, τ is the new independent variable, and

$$\bar{H}((p_1, \ldots, p_{N+1}), (q_1, \ldots, q_{N+1})) = H(p_{N+1}, q_{N+1}; q_1, \ldots, q_N) + \sum_{i=1}^{N} \sigma_i p_i. \tag{9}$$

Each of the first N equations in (7) reduces to $dt/d\tau = 1$. Then the first N equations in (8) reproduce Eq. 3, while the last of Eqs. 7 and 8 reproduce Eqs. 1 and 2. Note that each of the q_i, $i = 1, 2, \ldots N$, can be defined modulo 2π. The KAM theorem can be applied to the transformed system (7-9), thereby guaranteeing the existance of regular trajectories when $H(p, q, t)$ is sufficiently close to a time-independent Hamiltonian $H(p, q)$.

The system (7-9) has N integrals which are in involution. These are $q_i/\sigma_i - q_N/\sigma_N$, $i = 1, 2, \ldots N-1$, and $\bar{H}$. The integrals are independent provided $\partial H/\partial p \neq 0$. Only one additional integral is required to render the system integrable. The presence of the N integrals strongly constrains motion in the $2(N+1)$-dimensional phase space. It is useful to envision trajectories as curves which lie within a tube which densely covers an N-torus and whose cross-sectional coordinates are (p, q).

Numerical results which illustrate some general properties of this motion are shown in Fig. 1. In this example $N = 2$. The model system used to generate the results shown describes sound ray trajectories in a perturbed Munk (1974) model of the ocean sound channel,

$$\frac{c(z, r)}{c_o} = 1 + \epsilon(e^{\eta} - \eta - 1) +$$

$$\sum_{i=1}^{2} a_i \sin(i\pi e^{z/B}) \cos(2\pi r/\lambda_i + \varphi_i). \quad (10)$$

Here c is sound speed, z is depth, r is range, $\eta = 2(z - z_a)/B$ and and the constants were assigned the values $c_o = 1.49$ km/s, $\epsilon = 0.0057$, $B = 1.2$ km, $a_1 = a_2 = 0.0015$, $\varphi_1 = 0$, $\varphi_2 = 2.7$, $\lambda_1 = 20$ km and $\lambda_2 = 10(\sqrt{5} - 1)$ km. The ray equations are (note that z plays the role of the generalized coordinate q which is conjugate to the vertical slowness p, r is the time-like variable and $2\pi/\lambda_i$ is the frequency σ_i)

$$\frac{dz}{dr} = \frac{\partial H}{\partial p}, \quad (11)$$

$$\frac{dp}{dr} = -\frac{\partial H}{\partial z}, \quad (12)$$

with

$$H(p, z, r) = -\sqrt{c^{-2}(z, r) - p^2}. \quad (13)$$

For this problem the transformed system (7-9) has three degrees of freedom and two integrals so trajectories lie on surfaces of dimension $2 \times 3 - 2 = 4$ in the $2 \times 3 = 6$ dimensional bounded phase space. A three-dimensional slice (whose coordinates are $(z, p, r \bmod \lambda_2)$) of this space can be constructed by viewing trajectories at integer multiples of λ_1. In this 3-d space chaotic trajectories fill volumes while regular trajectories lie on surfaces. A second slice (whose thickness is nonzero) can be taken by plotting only those points in (z, p) which satisfy $|r \bmod \lambda_2 - r_o| < \delta$. Fig. 1 was constructed in this fashion using $\delta = \lambda_2/100$. The distribution of points shown is interpreted in the same way that Poincaré sections for autonomous 2 degree of freedom systems are interpreted: sequences of points corresponding to chaotic and regular trajectories fill areas and lie on smooth curves, respectively. Some blurring is present, however, because $\delta > 0$. With this minor caveat, Fig. 1 is seen to have the same qualitative features that are commonly seen in Poincaré sections for autonomous 2 degree of freedom systems. In particular, phase space appears to be partitioned into nonintersecting regular and chaotic regions. Stated somewhat differently, there is no evidence of Arnold diffusion (Arnold, 1989; Chirikov, 1979). This is surprising inasmuch as the transformed system (7-9) corresponding to (11-13) has three bounded degrees of freedom; generic systems with three or more bounded degrees of freedom are known to exhibit Arnold diffusion.

We now show that for the class of problems treated here, Arnold diffusion cannot occur. Consider a system of the form (7-9) which is a small perturbation to a time-independent (integrable) system. Chaotic motion arises in the vicinity of those trajectories which satisfy the commensurability condition

$$m_1\sigma_1 + m_2\sigma_2 + \ldots + m_N\sigma_N + m_{N+1}\omega = 0 \quad (14)$$

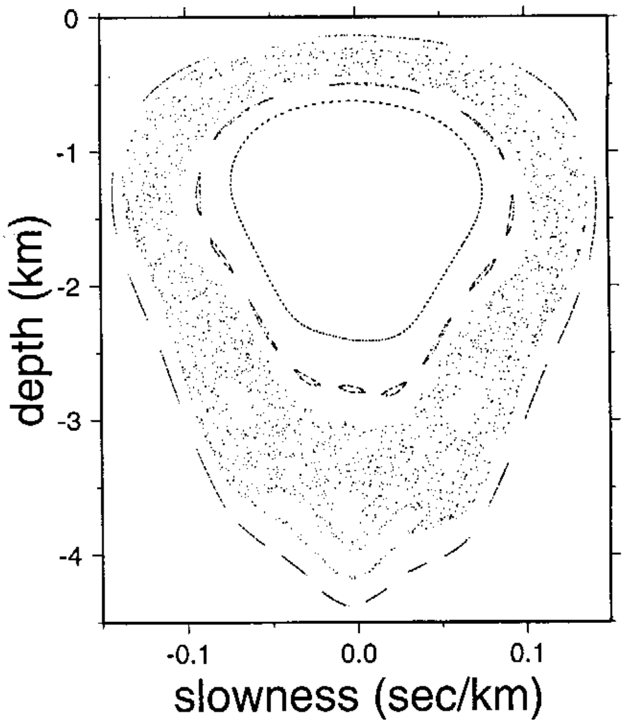

Fig. 1. Phase space portrait for a quasiperiodic Hamiltonian system which describes sound ray trajectories in a stratified ocean model perturbed by a superposition of $N = 2$ periodic functions of range, the time-like variable. Five trajectories are shown. This figure was constructed by twice slicing a bounded six-dimensional phase space as described in the text.

where the m_i's, $i = 1, 2, \ldots N + 1$ are nonzero integers and ω is the frequency of the unperturbed periodic motion in the (p, q) plane. Dividing (14) by σ_N gives

$$m_1\frac{\sigma_1}{\sigma_N} + m_2\frac{\sigma_2}{\sigma_N} + \quad \ldots \quad + m_{N-1}\frac{\sigma_{N-1}}{\sigma_N} + m_N + m_{N+1}\nu = 0 \quad (15)$$

where $\nu = \omega/\sigma_N$ is the winding number. Unlike ν, the ratios σ_i/σ_N, $i = 1, 2, \ldots N - 1$, are fixed properties of the environment. In contrast, in any fixed environment a continuum of ω's – and hence also ν's – will be present. This situation should be contrasted to that for autonomous $N + 1$ degree of freedom systems which, when perturbed, lose all of their integrals except for the Hamiltonian. For systems of the latter type Eq. 15 is replaced by

$$l_1\nu_1 + l_2\nu_2 + \ldots + l_N\nu_N + l_{N+1} = 0 \quad (16)$$

where the l_i's are nonzero integers. Solutions to Eq. 16, corresponding to different values of the l_i's form a web in ν-space. For $N \geq 2$, i.e., for three of more degrees of freedom, all points on this web are connected. This web – sometimes referred to as Arnold's web – connects all chaotic regions in ν-space. Chaotic trajectories may wander anywhere on this web via the process known as Arnold diffusion; see Chirikov (1979) for more details.

In contrast, in systems described by Eq. 16 with $N = 1$ (corresponding to two bounded degrees of freedom) or systems described by Eq. 15, only a 1-dimensional ν-space can be explored. For such systems the connectedness of solutions to Eq. 15 or Eq. 16 is lost; chaotic trajectories are isolated from each other in ν-space by bands of regular motion. The diffusion rate tends to increase as a system is perturbed away from an integrable state. Thus, Arnold diffusion is generally easy to detect numerically in systems that are far from integrable, but may be difficult to dectect in systems that are close to integrable. In spite of this minor caveat, it is clear that systems which do not exhibit Arnold diffusion are more strongly constrained – and fundamentally different – than those that do. The class of systems considered in this paper are of the former (constrained) type.

3 Fractal properties of trajectories

Discrete samples, $p(t_i), q(t_i), i = 1, 2, \ldots$, of the solutions to the equations of motion (1 and 2) may exhibit fractal properties, even when $H(p, q, t)$ is specified analytically. (This is somewhat surprising inasmuch as the functions $p(t)$ and $q(t)$ are infinitely differentiable when $H(p, q, t)$ is specified analytically.) Fractal properties of trajectories in Hamiltonian systems have recently been explored by Klafter et al. (1996); Osborne and Caponio (1990); Pasmanter (1988); Shlesinger et al. (1993); Zaslavsky et al. (1997). Most of this work has focused on autonomous two degree of freedom systems with a bounded phase space – or the equivalent class of area-preserving mappings. In such systems it has been shown that in systems with a mixed phase space, those trajectories which fill the chaotic seas have fractal properties. For nontrapped trajectories a commonly explored manifestation of fractal behavior is anomalous diffusion, i.e., rms growth of particle displacements proportional to $t^{1/D}$ where the fractal dimension D lies between 1 (ballistic motion) and 2 (Brownian motion). As the measure of regular regions of phase space approaches zero, D approaches 2. Because, as was shown above, the dynamics of $1\frac{1}{2}$ degree of freedom Hamiltonian systems whose time-dependence is quasiperiodic can be reduced to those of an area-preserving mapping, the mechanism that leads to fractal trajectories in systems of the latter type must also apply to systems of the former type. In the remainder of this section, this expectation is shown to be consistent with a numerical test and is discussed in the context of oceanic fluid parcel trajectories and problems involving ray dynamics.

A natural and robust means to characterize the fractal nature of trapped chaotic trajectories — such as those shown in Fig. 1 — is to make use of a box counting algorithm, based on the scaling relationship

$$C(s) \equiv \sum_i [P_i(s)]^2 \approx s^D. \quad (17)$$

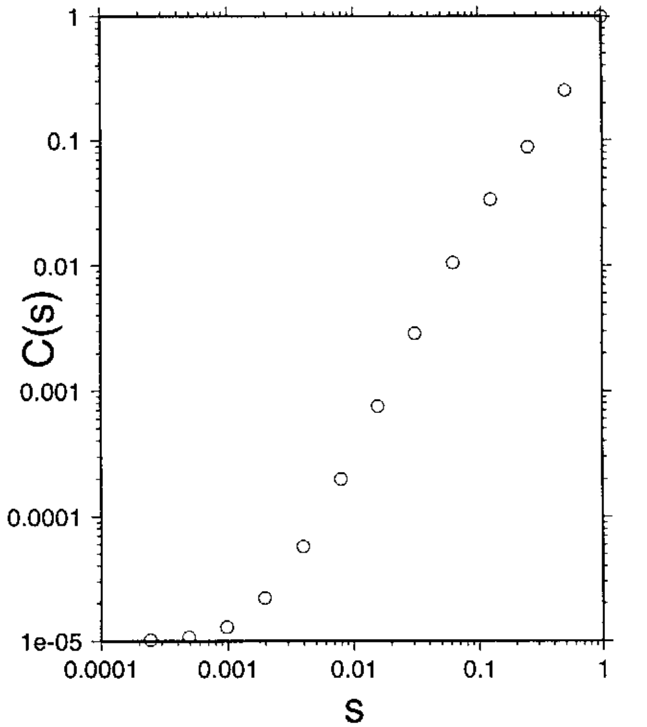

Fig. 2. The distribution function $C(s)$ vs. s for one of the trajectories which lies in the chaotic sea shown in Fig. 1, computed using a discretely sampled trajectory containing $M = 10^5$ points. For small s, $C(s)$ approaches the limit $1/M = 10^{-5}$. A least squares fit to the 10 rightmost points gives the slope estimate $D = 1.73 \pm 0.03$.

Here s^2 is the area of each box and $P_i(s)$ is the probability that a sample of the trajectory occupies the i'th box. Fractal behavior is accociated with $1 < D \leq 2$. The scaling relationship (17) was tested using one of the trajectories that fills the chaotic sea seen in Fig. 1. To compute $C(s)$ the domain shown in Fig. 1 was mapped onto the unit square with $s = 2^{-m}$, $m = 0, 1, 2, \ldots 12$ and $i = 1, 2, \ldots 2^{2m}$. The results are shown in Fig. 2; the fractal dimension $D \approx 1.73$. This estimate was found to be insensitive to the sampling interval. This simple example illustrates the result that the recently discovered connection between fractal properties of trajectories and chaotic Hamiltonian dynamics in systems with a mixed phase space extends to Hamiltonians with quasiperiodic time dependence. This connection provides a natural explanation for the occurrence of fractal trajectories in the two geophysical systems that we have discussed.

The fractal dimension of trajectories of surface drifters and submerged floats in mesoscale and large scale oceanic flows is $D \approx 1.3$ (Brown and Smith, 1990; Osborne et al., 1989). These trajectories $(x(t), y(t))$ satisfy equations of the form

$$\frac{dy}{dt} = \frac{\partial \psi}{\partial x}, \frac{dx}{dt} = -\frac{\partial \psi}{\partial y}.$$

Here the streamfunction $\psi(x, y, t)$ takes the place of the Hamiltonian $H(p, q, t)$ in Eqs. 1 and 2. While most oceanic fluid parcel trajectories appear to be chaotic and have fractal properties, there is also evidence of non-chaotic trajectories (see, e.g., Richardson et al., 1989) and hence evidence of the existance of regular islands in phase space (x, y). The simultaneous existance of regular and chaotic regions in phase space is, of course, consistent with the observation that fractal trajectories and a mixed phase space are closely linked. Thus, we interpret observations of apparently fractal oceanic drifter and float trajectories as an indication that the underlying Lagrangian ocean dynamics are those of a $1\frac{1}{2}$ degree of freedom Hamiltonian system with quasiperiodic time dependence and a mixed phase space. It is important to note that most oceanic flows can be realistically modelled as having quasiperiodic time dependence (possibly with large N), but cannot be realistically modelled as having periodic time dependence. (Periodic time dependence with a period which exceeds the duration of observations can be assumed, but such a picture is not useful from a dynamical systems point of view because regions of phase space are then not sampled more than once.) Although the conceptual picture of an oceanic streamfunction with quasiperiodic time-dependence is extremely useful for finite duration T observations, this picture cannot be expected to apply in the limit $T \to \infty$; in this limit the number N of periodic components of $\psi(x, y, t)$ required to reproduce fluid parcel trajectories is expected to increase without limit.

The conceptual picture of oceanic flows having quasiperiodic time dependence can be applied to flows which are characterized by power law energy spectra provided the number of spectral components in the description of the flow is finite. If the streamfunction for such a flow were known, it would be possible to construct a Poincaré section and identify regular and chaotic regions in the flow. In principle, Poincaré sections can be constructed using the procedure described above for arbitrary large (but finite) N; in practice, this procedure is feasible only for small N.

The application of our results to problems involving ray dynamics is more difficult to test because ray trajectories — and hence also their fractal dimension – are not directly measurable. In spite of this, Zaslavsky and Abdullaev (1997) have argued that the phenomenon of 'chaotic transmission' – whose origin is the nonuniformity of phase space and the stickiness of islands – should be measurable. For problems involving ray dynamics in environments with weak but complicated (describable as quasiperiodic with finite N) dependence on the time-like variable (range), we believe that the fractal nature of ray trajectories is less important than the stabilizing influence of regular regions in phase space. (Recall, however, that fractal trajectories and the existance of islands are related.) To understand the stabilizing influence of regular islands, note that under typical experimental conditions (for a point source, for example) the wavefield can be described as a sum of contributions from a continuum of rays; the existance of even a small number of regular regions within such a ray continuum constrains the motion of all the rays which make up the continuum and hence the entire wavefield.

4 Summary and discussion

In this paper we have considered nonautonomous 1 degree of freedom Hamiltonian systems for which H depends on N periodic functions of t. It was shown that such systems can be transformed to an autonomous $N+1$ degree of freedom system with a bounded phase space which has N integrals. The KAM theorem can be applied to the transformed system, guaranteeing that for a sufficiently small perturbation to an integrable system some regular motion is preserved. Furthermore, it was shown that when chaotic motion is present in such systems, Arnold diffusion cannot take place. Thus, the partitioning of phase space into nonintersecting regular and chaotic regions that characterizes the $N = 1$ (periodic) problem also applies to the quasiperiodic $N \geq 2$ problem. In other words, the dynamics of the latter class of problems can be reduced to those of an area-preserving mapping.

It has recently been shown that the partitioning of phase space into nonintersecting regular and chaotic regions that characterizes area-preserving mappings is closely linked to fractal behavior of trajectories which fill the chaotic seas. Thus, it follows from our results that the mechanisms that lead to fractal trajectories in area-preserving mappings also apply to 1 degree of freedom Hamiltonian systems whose time dependence is quasiperiodic. The geophysical systems that we have discussed can be realistically modelled as quasiperiodic systems. The results presented provide two important pieces of information about such systems: 1) in near-integrable systems some regular islands are expected; and 2) the occurrence of trajectories exhibiting fractal behavior in such systems is tied to the existance of regular islands in phase space.

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