

Another look at climate sensitivity

I. Zaliapin¹ and M. Ghil^{2,3}

¹Department of Mathematics and Statistics, University of Nevada, Reno, USA

²Geosciences Department and Laboratoire de Météorologie Dynamique (CNRS and IPSL), Ecole Normale Supérieure, Paris, France

³Department of Atmospheric & Oceanic Sciences and Institute of Geophysics & Planetary Physics, University of California, Los Angeles, USA

Received: 11 August 2009 – Revised: 15 January 2010 – Accepted: 24 February 2010 – Published: 17 March 2010

Abstract. We revisit a recent claim that the Earth’s climate system is characterized by sensitive dependence to parameters; in particular, that the system exhibits an asymmetric, large-amplitude response to normally distributed feedback forcing. Such a response would imply irreducible uncertainty in climate change predictions and thus have notable implications for climate science and climate-related policy making. We show that equilibrium climate sensitivity in all generality does not support such an intrinsic indeterminacy; the latter appears only in essentially linear systems. The main flaw in the analysis that led to this claim is inappropriate linearization of an intrinsically nonlinear model; there is no room for physical interpretations or policy conclusions based on this mathematical error. Sensitive dependence nonetheless does exist in the climate system, as well as in climate models – albeit in a very different sense from the one claimed in the linear work under scrutiny – and we illustrate it using a classical energy balance model (EBM) with nonlinear feedbacks. EBMs exhibit two saddle-node bifurcations, more recently called “tipping points,” which give rise to three distinct steady-state climates, two of which are stable. Such bistable behavior is, furthermore, supported by results from more realistic, nonequilibrium climate models. In a truly nonlinear setting, indeterminacy in the size of the response is observed only in the vicinity of tipping points. We show, in fact, that small disturbances cannot result in a large-amplitude response, unless the system is at or near such a point. We discuss briefly how the distance to the bifurcation may be related to the strength of Earth’s ice-albedo feedback.

1 Introduction and motivation

1.1 Climate sensitivity and its implications

Systems with feedbacks are an efficient mathematical tool for modeling a wide range of natural phenomena; Earth’s climate is one of the most prominent examples. Stability and sensitivity of feedback models is, accordingly, a traditional topic of theoretical climate studies (Cess, 1976; Ghil, 1976; Crafoord and Källén, 1978; Schlesinger, 1985, 1986; Cess et al., 1989). Roe and Baker (2007) (RB07 hereafter) have recently advocated existence of intrinsically large sensitivities in an equilibrium model with multiple feedbacks. Specifically, they argued that a small, normally distributed feedback may lead to large-magnitude, asymmetrically distributed values of the system’s response.

Such a property, if valid, would have serious implications for climate dynamics (Allen and Frame, 2007) and for modeling of complex systems in general (Watkins and Freeman, 2008). In this paper, we revisit the dynamical behavior of a general, equilibrium climate model with genuinely nonlinear feedbacks, and focus subsequently on a simple energy-balance model (EBM). We notice that the main, technical part of RB07’s argument is well-known in the climate literature, cf. Schlesinger (1985, 1986), and thus it seems useful to review the associated assumptions and possible interpretations of this result.

We rederive below in Sect. 1.2 the key equation of RB07 and comment on their purportedly nonlinear analysis in Sect. 1.3. We then proceed in Sect. 2 with a more self-consistent version of sensitivity analysis for a nonlinear model. This analysis is applied in Sect. 3 to a zero-dimensional EBM. Concluding remarks follow in Sect. 4.



Correspondence to: I. Zaliapin
(zal@unr.edu)

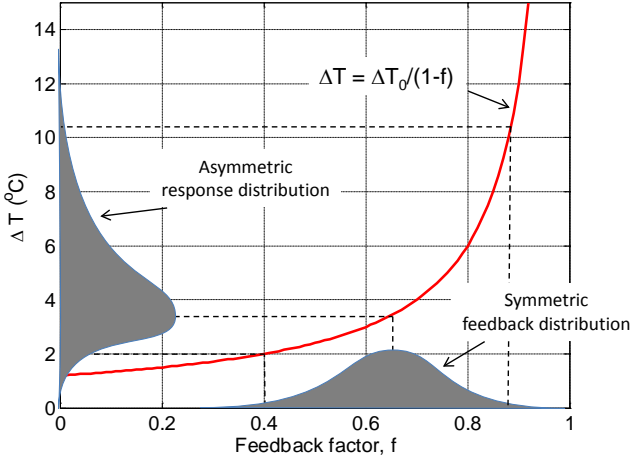


Fig. 1. Temperature response ΔT as a function of the feedback factor f in a linear climate model; modified after Roe and Baker (2007). In the absence of feedbacks ($f = 0$), the response is given by $\Delta T = \Delta T_0 = 1.2^\circ\text{C}$. The response is amplified by feedbacks; it diverges ($\Delta T \rightarrow \infty$) as $f \rightarrow 1$. The shaded areas illustrate the hypothetical symmetric distribution of the feedback factor f and the corresponding asymmetric distribution of the system response ΔT . Our study shows that this prominently asymmetric response is only seen in a linear model and is absent in a general nonlinear model (see Fig. 2 below).

1.2 Roe and Baker's (2007) linear analysis

We follow here RB07 and assume the following general setup. Let the net radiation R at the top of the atmosphere be related to the corresponding average temperature T at the Earth's surface by $R = R(T)$. Assume, furthermore, that there exists a feedback $\alpha = \alpha(T)$, which is affected by the temperature change and which can, in turn, affect the radiative balance. Hence, one can write $R = R(T, \alpha(T))$.

To study how a small change ΔR in the radiation is related to the corresponding temperature change ΔT , one can use the Taylor expansion (Arfken, 1985) to obtain, as ΔT tends to zero,

$$\Delta R = \frac{\partial R}{\partial T} \Delta T + \frac{\partial R}{\partial \alpha} \frac{\partial \alpha}{\partial T} \Delta T + \mathcal{O}((\Delta T)^2). \quad (1)$$

Here, $\mathcal{O}(x)$ is a function such that $|\mathcal{O}(x)| \leq Cx$ as soon as $0 < x < \epsilon$ for some positive constants C and ϵ .

Introducing the notations

$$\frac{1}{\lambda_0} = \frac{\partial R}{\partial T}, \quad \text{and} \quad f = -\lambda_0 \frac{\partial R}{\partial \alpha} \frac{\partial \alpha}{\partial T},$$

for the "reference sensitivity" λ_0 and the "feedback factor" f , we obtain

$$\Delta R = \frac{1-f}{\lambda_0} \Delta T + \mathcal{O}((\Delta T)^2), \quad (2)$$

which readily leads to

$$\Delta T = \frac{\lambda_0}{1-f} \Delta R + \mathcal{O}((\Delta T)^2), \quad (3)$$

as long as $f \neq 1$.

RB07 drop the higher-order terms in Eq. (3) to obtain

$$\Delta T = \frac{\lambda_0}{1-f} \Delta R, \quad (4)$$

which is their Eq. (S4). This equation leads directly to their main conclusion, namely that a normally distributed feedback factor f results in an asymmetric system response ΔT to a fixed forcing ΔR . The purported sensitivity is due to the divergence of the right-hand side (rhs) of Eq. (4) (their Eq. S4) as f approaches unity. Figure 1, which is analogous to Fig. 1 of RB07, illustrates this effect.

Roughly speaking, RB07 use the following argument: If the derivative of $R(T)$ with respect to T is close to 0, then the derivative of T with respect to R is very large, and a small change in the radiation R corresponds to a large change in the temperature T . Such an argument, though, is only valid for an essentially linear dependence $R \propto T$. Our straightforward analysis in Sect. 2 below shows that the sensitivity effect of Fig. 1 is absent in climate models in which genuinely nonlinear feedbacks $R = R(T, \alpha(T))$ are present.

It is worth noticing that, since one seeks the temperature change ΔT that results from a change ΔR in the forcing, it might be preferable to consider the inverse function $T = T(R)$ or, more precisely, $T = T(R, \alpha(R))$ and the corresponding Taylor expansion

$$\Delta T = \frac{\partial T}{\partial R} \Delta R + \frac{\partial T}{\partial \alpha} \frac{\partial \alpha}{\partial R} \Delta R + \mathcal{O}((\Delta R)^2).$$

The main conclusions of our analysis will not be affected by the particular choice of direct or inverse expansion, provided the nonlinearities are correctly taken into account.

1.3 Roe and Baker's "nonlinear" analysis

In this section we address the analysis carried out by RB07 in the supplemental on-line materials, pp. 4–5, Sect. "Non-linear feedbacks." The main conclusion of that analysis was that, given realistic parameter values for the climate system, the effects of possible nonlinearities in the behavior of the function $R = R(T)$ are negligible and do not affect the system's sensitivity. We point out here two serious flaws in their mathematical reasoning that, each separately and the two together, invalidate such a conclusion.

First, and most importantly, despite their section's title, the analysis carried out by RB07 is still *linear*. Indeed, the Taylor expansion in their Eq. (S7) is given by

$$\Delta R \approx R' \Delta T + \frac{1}{2} R'' \Delta T^2, \quad (5)$$

where $(\cdot)'$ stands for differentiation with respect to T . But RB07 immediately invert this equation for ΔT subject to the assumption

$$\Delta T^2 = \Delta T \Delta T_0,$$

where ΔT_0 is a constant. Hence, instead of solving the quadratic Eq. (5), Roe and Baker solve the following *linear* approximation:

$$\Delta R = \left(R' + \frac{1}{2} R'' \Delta T_0 \right) \Delta T,$$

and thus obtain the key formula (their Eq. S8)

$$\Delta T \approx \frac{-\lambda_0 \Delta R}{1 - f - \frac{1}{2} \lambda_0 R'' \Delta T_0}; \quad (6)$$

the last step uses the, correct, fact that $R' = (1 - f)/\lambda_0$. This equation artificially introduces a divergence point for the temperature at $f = 1 - (\lambda_0/2) R'' \Delta T_0$, which clearly cannot exist in a quadratic equation. Equation (6) is thus a very crude approximation that significantly deviates from the true solution to the full quadratic Eq. (5) – which we discuss below in Sect. 2 – and thus cannot be used to justify general statements about climate models.

The second flaw in the Roe and Baker (2007) reasoning is that, using the model and parameter values they suggest, one readily finds that:

- $R' = -2$, i.e., global temperature and radiation are *negatively* correlated, which is hardly the case for the current climate (e.g., Held and Soden, 2000). We notice that the negative sign of the correlation follows directly from their Eq. (S10) and is not affected by particular values of the model parameters. Furthermore,
- $R'' = -0.03$, which means that the model they consider is, indeed, essentially *linear*, and thus not very realistic.

Although, in this part of their analysis, Roe and Baker assumed that $f = 0.4$, it is easy to check that, for all $f < 0.95$, their model satisfies $|R''/R'| < 0.1$ and is therewith very close to being linear. To conclude, the effects on nonlinearities are indeed negligible in the particular model studied in this part of the RB07 paper, since the model is very close to being linear; one cannot extrapolate, therefore, their conclusions to climate models with significant nonlinearities.

We next proceed with a mathematically correct sensitivity analysis of a general climate model in the presence of truly nonlinear feedbacks.

2 A self-consistent sensitivity analysis

It is easily seen from the discussion in Sect. 1.2, especially from Eq. (2), that the relationship (4) is a crude approximation: it is valid only subject to the assumptions that (a) the

higher-order terms in the expansion of ΔT are vanishingly small:

$$\mathcal{O}((\Delta T)^2) \ll \frac{1-f}{\lambda_0} \Delta T;$$

and (b) the quantity in the rhs of this inequality is itself nonzero.

If one assumes, for instance, that $\mathcal{O}((\Delta T)^2) \sim C(\Delta T)^2$, where the precise meaning of $g(x) \sim f(x)$ is given by $\lim_{x \rightarrow 0} g(x)/f(x) = 1$, then the assumptions (a, b) above hold for ΔT that satisfy both of the following conditions

$$0 < \Delta T \ll (1-f)/(\lambda_0 C) \quad \text{and} \quad 0 < \Delta T < \epsilon, \quad (7)$$

where C and ϵ are defined after Eq. (1). The first of these conditions implies that the range of temperatures within which the approximation (4) works vanishes as the feedback factor f approaches unity. Hence, all the results based on this approximation – including precisely the main conclusions of RB07 – no longer apply outside a vanishingly small neighborhood of $f = 1.0$.

The asymptotic behavior we assumed above for $\mathcal{O}((\Delta T)^2)$ is not exotic. Consider for instance the function $R = T^2$ in the neighborhood of $R = 0$. Its Taylor expansion

$$\Delta R = 2T \Delta T + \mathcal{O}((\Delta T)^2)$$

can be used to obtain, ignoring the second-order term,

$$\Delta T \approx \Delta R / (2T). \quad (8)$$

The last equation would seem to imply that the growth of ΔT is inversely proportional to T itself, so the change in T should increase infinitely fast as T goes to 0, a rather annoying contradiction.

The way out of this conundrum is to realize that the change ΔT given by Eq. (8) is only valid in a small vicinity of $T = 0$ and cannot be extrapolated to larger values. Of course, we all know that the function $R = T^2$ is nicely bounded and smooth in the vicinity of 0, but it is essential to take into account the second term in its Taylor expansion in this vicinity to obtain correct results. We show in Sect. 3 below that this simple example depicts the essential dependence of the Earth surface temperature on the global solar radiative input, for conditions close to those of the current Earth system.

In summary, the linear approximation of the function $R(T)$ derived by RB07 from its Taylor expansion is not valid when f approaches unity. In this case – which is precisely the situation emphasized by these authors – the higher-order terms “hidden” inside $\mathcal{O}((\Delta T)^2)$, which they neglected, are indispensable for a correct, self-consistent climate sensitivity analysis.

A correct analysis of the case when f approaches unity needs to start with a Taylor expansion that keeps the second-order term

$$\Delta R = \frac{1-f}{\lambda_0} \Delta T + a(\Delta T)^2 + \mathcal{O}((\Delta T)^3),$$

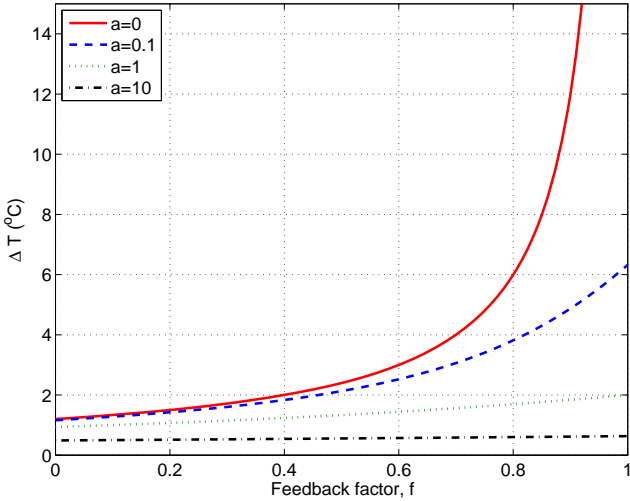


Fig. 2. Temperature response ΔT as a function of feedback factor f in a nonlinear, quadratic climate model, governed by Eq. (9). The curves correspond (from top to bottom) to increasing values of the magnitude $a = R''/2$ of the quadratic term; the values of a for each curve are given in the figure's legend, in the upper-left corner. The uppermost curve (red) corresponds to $a = 0$ and is the same as the one shown in Fig. 1 above. Extreme sensitivity, expressed in the divergence of ΔT , is only seen in the linear model; it rapidly vanishes in the nonlinear model, as the nonlinearity factor a increases.

where $a = R''/2$. If $\mathcal{O}((\Delta T)^3)$ is much smaller than the other two terms on the rhs, then the temperature change can be approximated by a solution of the quadratic equation

$$\frac{1-f}{\lambda_0} \Delta T + a(\Delta T)^2 = \Delta R. \quad (9)$$

The real-valued solutions to the latter equation, if they exist, are given by

$$\Delta T_{1,2} = \frac{1}{2} \left(\frac{f-1}{a\lambda_0} \pm \sqrt{\left(\frac{1-f}{a\lambda_0} \right)^2 + \frac{4\Delta R}{a}} \right).$$

In particular, when f is close to 1.0, then

$$\Delta T_{1,2} \approx \pm \sqrt{\frac{\Delta R}{a}}. \quad (10)$$

One can see from Eq. (10) that the proximity of the feedback factor f to unity no longer plays an important role in the qualitative behavior of the equilibrium temperature. This point is further illustrated in Fig. 2 that shows the climate system's response ΔT as a function of the feedback factor f for different values of the nonlinearity parameter a . The most important observation is that the climate response does not diverge at $f = 1$; moreover, the asymmetry of the response due to the changes in feedback factor f rapidly vanishes as soon as the dependence of ΔR on ΔT becomes nonlinear.

In general, one can consider an arbitrary number of terms in the Taylor expansion of $R(T)$. The very fact that one relies on the validity of the Taylor expansion implies that $R(T)$ is bounded and sufficiently smooth; in other words, a divergence of the equilibrium temperature due to a small change in the forcing contradicts the very assumptions on which RB07 based their sensitivity analysis.

3 Sensitivity for energy balance models (EBMs)

We consider here a classical climate model with nonlinear feedbacks to illustrate that, in such a model: (i) the type of sensitivity claimed by RB07 does not exist; and (ii) sensitive dependence may exist, in a very different sense, namely in the neighborhood of bifurcation points, as explained below.

3.1 Model formulation

We consider here a highly idealized type of model that connects the Earth's temperature field to the solar radiative flux. The key idea on which these models are built is due to Budyko (1969) and Sellers (1969). They have been subsequently generalized and used for many studies of climate stability and sensitivity; see Held and Suarez (1974); North (1975) and Ghil (1976), among others.

The interest and usefulness of these “toy” models resides in two complementary features: (i) their simplicity, which allows a complete and thorough understanding of the key mechanisms involved; and (ii) the fact that their conclusions have been extensively confirmed by studies using much more detailed and presumably realistic models, including general circulation models (GCMs); see, for instance, the reviews of North et al. (1981) and Ghil and Childress (1987).

The main assumption of EBMs is that the rate of change of the global average temperature T is determined only by the net balance between the absorbed radiation R_i and emitted radiation R_o :

$$c \frac{dT}{dt} = R_i(T) - R_o(T). \quad (11)$$

For simplicity, we follow here the zero-dimensional (0-D) EBM version of Crafoord and Källén (1978) and Ghil and Childress (1987), in which only global, coordinate-independent quantities enter; thus

$$R_i = \mu Q_0(1 - \alpha(T)), \quad R_o = \sigma g(T)T^4. \quad (12)$$

In the present formulation, the planetary *ice-albedo feedback* α decreases in an approximately linear fashion with T , within an intermediate range of temperatures, and is nearly constant for large and small T . Here Q_0 is the reference value of the global mean solar radiative input, σ is the Stefan-Boltzmann constant, and $g(T)$ is the grayness of the Earth, i.e. its deviation from black-body radiation σT^4 . The parameter $\mu \approx 1.0$ multiplying Q_0 indicates by how much the global insolation deviates from its reference value.

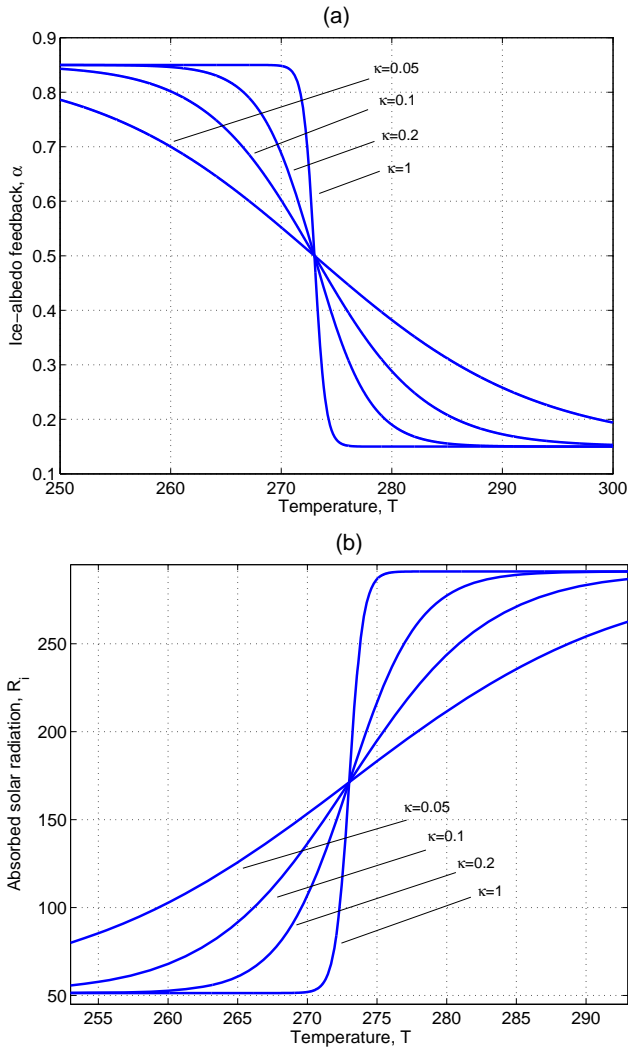


Fig. 3. Dependence of the absorbed incoming radiation R_i on the steepness parameter κ : (a) ice-albedo feedback $\alpha = \alpha(T; \kappa)$, and (b) absorbed radiation $R_i = R_i(T; \kappa)$, for different values of κ ; see Eqs. (12) and (13).

We model the ice-albedo feedback by

$$\alpha(T; \kappa) = c_1 + c_2 \frac{1 - \tanh(\kappa(T - T_c))}{2}. \quad (13)$$

This parametrization represents a smooth interpolation between the piecewise-linear formula of Sellers-type models, like those of Ghil (1976) or Crafoord and Källén (1978), and the piecewise-constant formula of Budyko-type models, like those of Held and Suarez (1974) or North (1975).

Figure 3a shows four profiles of our ice-albedo feedback $\alpha(T) = \alpha(T)$ as a function of T , depending on the value of the steepness parameter κ . The profile for $\kappa \gg 1$ would correspond roughly to a Budyko-type model, in which the albedo α takes only two constant values, high and low, depending on whether $T < T_c$ or $T > T_c$. The other profiles

shown in the figure for smaller values of κ , correspond to Sellers-type models, in which there exists a transition ramp between the high and low albedo values. Figure 3b shows the corresponding shapes of the radiative input $R_i = R_i(T)$.

The *greenhouse effect* is parametrized, as in Crafoord and Källén (1978) and Ghil and Childress (1987), by letting

$$g(T) = 1 - m \tanh\left(\frac{T}{T_0}\right). \quad (14)$$

Substituting this greenhouse effect parametrization and the one for the albedo into Eq. (11) leads to the following EBM:

$$c\dot{T} = \mu Q_0(1 - \alpha(T)) - \sigma T^4 \left[1 - m \tanh\left(\frac{T}{T_0}\right)\right], \quad (15)$$

where $\dot{T} = dT/dt$ denotes the derivative of global temperature T with respect to time t .

It is important to note that current concern, both scientific and public, is mostly with the greenhouse effect, rather than with actual changes in insolation. But in a simple EBM model – whether globally averaged, like in Crafoord and Källén (1978) and here, or coordinate-dependent, as in Budyko (1969); Sellers (1969); Held and Suarez (1974); North (1975) or Ghil (1976) – increasing μ always results in a net increase in the radiation balance. It is thus convenient, and quite sufficient for the purpose at hand, to vary μ in the incoming radiation R_i , rather than some other parameter in the outgoing radiation R_o . We shall return to this point in Sect. 4.

3.2 Model parameters

The value S of the *solar constant*, which is the value of the solar flux normally incident at the top of the atmosphere along a straight line connecting the Earth and the Sun, is assumed here to be $S = 1370 \text{ W m}^{-2}$. The reference value of the global mean solar radiative input is $Q_0 = S/4 = 342.5$, with the factor 1/4 due to Earth's sphericity.

The parameterization of the ice-albedo feedback in Eq. (13) assumes $T_c = 273 \text{ K}$ and $c_1 = 0.15$, $c_2 = 0.7$, which ensures that $\alpha(T)$ is bounded between 0.15 and 0.85, as in Ghil (1976); see Fig. 3a. The greenhouse effect parametrization in Eq. (14) uses $m = 0.4$, which corresponds to 40% cloud cover, and $T_0^{-6} = 1.9 \times 10^{-15} \text{ K}^{-6}$ (Sellers, 1969; Ghil, 1976). The Stefan-Boltzmann constant is $\sigma \approx 5.6697 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$.

3.3 Sensitivity and bifurcation analysis

3.3.1 Two types of sensitivity analysis

We distinguish here between two types of sensitivity analysis for the 0-D EBM (11). In the first type, we assume that the system is driven out of an equilibrium state $T = T_0$, for which $R_i(T_0) - R_o(T_0) = 0$, by an external force, and want to see whether and how it will return to a new equilibrium state,

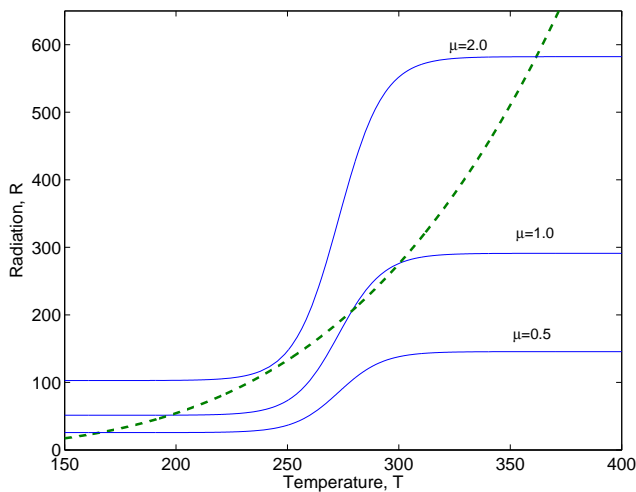


Fig. 4. Outgoing radiation R_o (red, dashed line) and absorbed incoming solar radiation R_i (blue, solid lines) for our 0-D energy-balance model (EBM), governed by Eq. (15). The absorbed radiation is shown for $\mu = 0.5, 1$ and 2.0 (from bottom to top), while $\kappa = 0.05$. Notice the existence of one or three intersection points between the R_o curve and one of the R_i curves, depending on the value of μ ; these points correspond to the equilibrium solutions of Eq. (15), i.e. to steady-state climates.

which may be different from the original one. This analysis refers to the “fast” dynamics of the system, and assumes that $R_i(T) - R_o(T) \neq 0$ for $T \neq T_0$; it is often referred to as *linear stability analysis*, since it considers mainly small displacements from equilibrium at $t = 0$, $T(0) = T_0 + \theta(0)$, where $\theta(0)$ is of order ϵ , with $0 < \epsilon \ll 1$, as defined in Sect. 2.

The second type of analysis refers to the system’s “slow” dynamics. We are interested in how the system evolves along a *branch of equilibrium solutions* as the external force changes sufficiently slowly for the system to track an equilibrium state; hence, this second type of analysis always assumes that the solution is in equilibrium with the forcing: $R_i(T) - R_o(T) = 0$ for all T of interest. Typically, we want to know how sensitive model solutions are to such a slow change in a given parameter, and so this type of analysis is called *sensitivity analysis*. In the problem at hand, we will study – again following Crafoord and Källén (1978) and Ghil and Childress (1987) – how changes in μ , and hence in the global insolation, affect the model’s equilibria.

A remarkable property of the EBM governed by Eq. (11) is the existence of several stationary solutions that describe equilibrium climates of the Earth (Ghil and Childress, 1987). The existence and linear stability of these solutions result from a straightforward *bifurcation* analysis of the 0-D EBM (11), as well as of its one-dimensional, latitude-dependent counterparts (Ghil, 1976, 1994): there are two linearly stable solutions – one that corresponds to the present climate and one that corresponds to a much colder, “snowball Earth” (Hoffman et al., 1998) – separated by an unstable one, which lies about 10 K below the present climate.

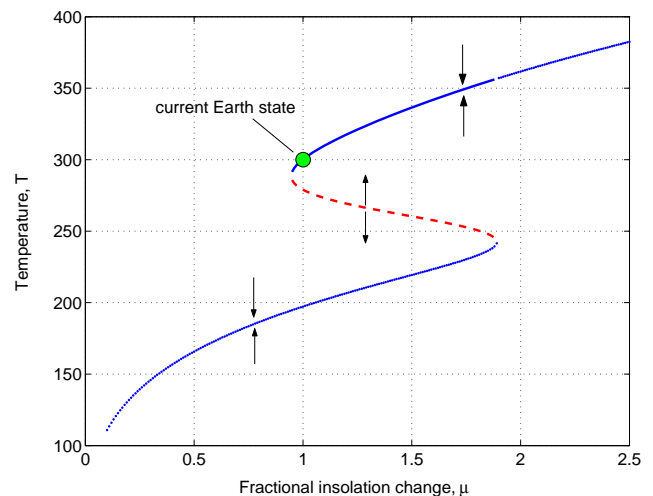


Fig. 5. Equilibrium solutions of the EBM (15) depending on the fractional change μ in insolation. Notice the existence of two stable (blue, solid lines) and one unstable (red, dashed line) solution branches. The arrows show the direction in which the global temperature will change after being displaced from a nearby equilibrium state by external forces. The current Earth state corresponds to the upper stable solution at $\mu = 1$; in this figure $\kappa = 0.05$.

The existence of the three equilibria – two stable and one unstable – has been confirmed by such results being obtained by several distinct EBMs, of either Budyko- or Sellers-type (North et al., 1981; Ghil, 1994). Nonlinear stability, to large perturbations in the initial state, has been investigated by introducing a variational principle for the latitude-dependent EBMs of Sellers (Ghil, 1976) and of Budyko (North et al., 1981) type, and it confirms the linear stability results.

3.3.2 Sensitivity analysis for a 0-D EBM

We analyze here the stability of the “slow,” quasi-adiabatic (in the statistical-physics sense) dynamics of model (15). The energy-balance condition for steady-state solutions $R_i = R_o$ takes the form

$$\mu Q_0(1 - \alpha(T)) = \sigma T^4 \left[1 - m \tanh\left(\left(\frac{T}{T_0}\right)^6\right) \right]. \quad (16)$$

We assume here, following the previously cited EBM work, that the main bifurcation parameter is μ ; this happens to agree with the emphasis of Roe and Baker (2007) on climate sensitivity as the dependence of mean temperature T on global solar radiative input, denoted here by $Q = \mu Q_0$.

Figure 4 shows the absorbed and emitted radiative fluxes, R_i and R_o , as functions of temperature T for $\mu = 0.5, 1$ and 2.0 . One can see that Eq. (16) may have one or three solutions depending on the value of μ : only the present, relatively warm climate for $\mu = 2.0$, only the “deep-freeze” climate for $\mu = 0.5$, and all three, including the intermediate, unstable one for present-day insolation values, $\mu = 1.0$. These

steady-state climate values are shown as a function of the insolation parameter μ in the bifurcation diagram of Fig. 5.

The “fast” stability analysis (not presented here) shows that small deviations $\theta(0)$ from an equilibrium solution, while all parameter values are kept fixed, may result in two types of dynamics, depending on the initial equilibrium T_0 : fast increase or fast decrease of the initial deviation (Ghil and Childress, 1987). The fast increase characterizes *unstable* equilibria: a small deviation $\theta(0)$ from such an equilibrium T_0 forces the solution to go further and further away from the equilibrium. In practice, such equilibria will not be observed, since there are always small, random perturbations of the climate present in the system: just think of weather as representing such perturbations.

The fast decrease of the initial deviation $\theta(0)$ characterizes *stable* solutions; only such equilibria can be observed in practice. The two stable solution branches of Eq. (15) are shown by solid lines in Fig. 5, while the unstable branch is shown by the dashed line. The arrows show the direction in which the temperature will change when drawn away from an equilibrium by external forces. This change, whether away from or towards the nearest equilibrium, is fast compared to the one that occurs along either solution branch (Ghil, 1976, 1994).

3.3.3 Bifurcation analysis

Given the choice of model parameters, the present climate state corresponds to the upper stable solution of Eq. (15), at $\mu = 1$ (see Fig. 5). It lies quite close to the bifurcation point $(\mu, T) \approx (0.9, 280 \text{ K})$, where the stable and unstable solutions merge.

The so-called *normal form* of this bifurcation is given by the equation

$$\dot{X} = \bar{\mu} - X^2, \quad (17)$$

where X is a suitably normalized form of T , and $\bar{\mu}$ is a normalized form of μ . Equation (17) describes the dependence between T and μ in a small neighborhood of the bifurcation point. In particular, the stable equilibrium branch is described by

$$X = +\sqrt{\bar{\mu}};$$

this result has exactly the same form as the positive solution of Eq. (10), given by our self-consistent analysis of climate sensitivity in the presence of genuine nonlinearities, cf. Sect. 2. Hence, the derivative $dX/d\bar{\mu}$, and thus $dT/d\mu$, goes to infinity as the model approaches the bifurcation point; this is exactly the situation discussed earlier in Sect. 2.

It is important to realize that the parabolic form of temperature dependence on insolation change is not an accident due to the particularly simple form of EBMs. Wetherald and Manabe (1975) clearly showed, in a slightly simplified GCM, that not only the mass-weighted temperature of their total atmosphere, but also the area-weighted temperatures of each

of their five model levels, exhibits such a parabolic dependence on fractional radiative input; see Fig. 5 in their paper. Moreover, these authors emphasize that “As stated in the Introduction, it is not, however, reasonable to conclude that the present results are more reliable than the results from the one-dimensional studies mentioned above simply because our model treats the effect of transport explicitly rather than by parameterization. [...] Nevertheless, it seems to be significant that both the one-dimensional and three-dimensional models yield qualitatively similar results in many respects.”

In fact, rigorous mathematical results demonstrate that the *saddle-node bifurcation* whose normal form is given by Eq. (17) occurs in several systems of nonlinear partial differential equations, such as the Navier-Stokes equations (Constantin et al., 1989; Temam, 1997), and not only in ordinary differential equations, like Eqs. (11) and (15) above. We emphasize, though, that this does not cause the temperature to increase rapidly due to small changes in insolation: the presence of the bifurcation point will result in small, positive changes of global temperature for slow, positive changes in μ , while it may throw the climate system into the deep-freeze state for slow, negative changes in μ .

4 Discussion

4.1 How sensitive is climate?

Making projections of climate change for the next decades and centuries, evaluating the human influence on future Earth temperatures, and making normative decisions about current and future anthropogenic impacts on climate are enormous tasks that require solid scientific expertise, as well as responsible moral reasoning. Well-founded approaches to handle the moral aspects of the problem are still being debated (e.g., Hillerbrand and Ghil, 2008). It is that much more important to master existing tools for acquiring accurate and reliable scientific evidence from the available data and models. Several of these tools come from the realm of nonlinear and complex dynamical systems (Lorenz, 1963; Smale, 1967; Ruelle and Takens, 1971; Ghil and Childress, 1987; Ghil, 1994; Ghil et al., 2008).

A straightforward analysis, carried out in Sect. 2 of this paper, shows that a proper treatment of the higher-order terms in a climate model with nonlinear feedbacks does not reveal the exaggerated sensitivity to forcing that was used in RB07 to advocate intrinsic unpredictability of climate projections. We emphasize that the error in Roe and Baker’s analysis is not related to their choice of model formulation or of the model parameters nor to their interpretation of model results. The problem is purely a matter of elementary calculus, and is due to inappropriate, and unnecessary, linearization of a nonlinear model.

Our analysis complements, reinforces and goes beyond that of Hannart et al. (2009), who also showed that the claim

of RB07 “results from a mathematical artifact.” We notice simply that Hannart and colleagues did not even question the linear approximation framework of RB07 and still concluded that the claims of irreducibility of the spread in the envelope of climate sensitivity are not supported by the RB07 analysis.

To summarize, while the general *human* concern about climate sensitivity expressed by RB07 should be reasonably shared by many, their *scientific* conclusions do not follow from their model and its results, when correctly analyzed, as done here in Sect. 2. Nor are these conclusions supported by other models of greater detail and realism, when properly investigated. Accordingly, conclusions about the likelihood of extreme warming resulting from small changes in anthropogenic forcing can hardly be used to support political proposals (e.g., Allen and Frame, 2007) that claim to provide future directions for the climate-related sciences.

Still, this paper’s analysis does not preclude in any sense the Earth’s temperature from rising significantly in coming years. The methods illustrated here can only be used to study climate sensitivity in the vicinity of a given state; they cannot be applied to investigate climate evolution over tens of years, for example in response to large increases in greenhouse gases or to other major changes in the forcing, whether natural or anthropogenic. This latter problem requires global interdisciplinary efforts and, in particular, the analysis of the entire hierarchy of climate models (Schneider and Dickinson, 1974), from conceptual to intermediate to fully coupled GCMs (Ghil and Robertson, 2000). It also requires a much more careful study of random effects than has been done heretofore (Ghil et al., 2008).

It seems to us that Roe and Baker’s title question “Why Is Climate Sensitivity So Unpredictable?” still remains open.

4.2 Where are the “tipping points”?

The S-shaped diagram of Fig. 5 – see also Fig. 10.6 in Ghil and Childress (1987) and Fig. 4 in Ghil (1994) – was used here to show the smoothness and boundedness of temperature changes as a function of insolation changes, away from a saddle-node bifurcation, like that of Eq. (10) in Sect. 2 or of Eq. (17) in Sect. 3.3.3.

This S-shaped curve nevertheless reveals the existence of sensitive dependence of Earth’s temperature on insolation changes, or on other changes in Earth’s net radiation budget, such as may be caused by increasing levels of greenhouse gases, on the one hand, or of aerosols, on the other. This sensitive dependence is quite different from the one advocated by RB07. Namely, if the parameter μ were to slightly decrease – rather than increase, as it seems to have done since the mid-1970s, in the sense described in the last paragraph of Sect. 3.1 – then the climate system would be pushed past the bifurcation point at $\mu \approx 0.9$. The only way for the global temperature to go would be down, all the way to a deep-freeze Earth, with much lower temperatures than those of recent, Quaternary ice ages.

It has become common in recent discourse about potentially irreversible climate change to talk about “tipping points”; e.g., Lenton et al. (2008). The term was originally introduced into the social sciences by Gladwell (2000) to denote a point at which a previously rare phenomenon becomes dramatically more common. In the physical sciences, it has been identified with a shift from one stable equilibrium to another one, i.e., to a saddle-node bifurcation, as seen in Fig. 5 here and explained in Sect. 3.3.3 above.

In the EBM context of Fig. 5, it would require an enormous, almost twofold increase in the insolation in order for a deep-freeze-type equilibrium to reach the bifurcation point at $\mu \approx 1.85$ and jump from there to $T \approx 350$ K, a temperature that sounds equally unpleasant. Within the broader context of the recent debates on how to exit a snowball-Earth state, very large, and possibly implausible increases in CO₂ levels would be required (Pierrehumbert, 2004).

Indeed, the likelihood to actually reach the tipping point to the left of the current climate in Fig. 5 seems to be quite small. Mechanisms for entering a snowball-Earth climate have been recently studied with a number of fairly realistic climate models (Hyde et al., 2000; Donnadieu et al., 2004; Poulsen and Jacob, 2004). Both modeling and independent geological evidence suggest that Earth’s climate can sustain significant fluctuations of the solar radiative input, and hence of global temperature, without entering the snowball Earth, and evidence for Earth ever having been in such a state is still controversial.

Nevertheless, the existence of the upper-left tipping point shown in Fig. 5 is confirmed by numerous model studies, including GCMs, and we have already cited some evidence also for the lower-right tipping point in the figure. Several hypothetical tipping points on the “warm” side have been identified by Lenton et al. (2008) and references therein, among many others. But only few of these have been studied with the same degree of mathematical and physical detail as the ones of Fig. 5 here. One worthwhile example is that of the oceans’ buoyancy-driven, or thermohaline, circulation (Stommel, 1961; Bryan, 1986; Quon and Ghil, 1992; Thual and McWilliams, 1992; Dijkstra and Ghil, 2005).

Accordingly, humankind must be careful – in pursuing its recent interest in geoengineering (Crutzen, 2006; MacCracken, 2006) – to stay a course that runs between tipping points on the warm, as well as on the “cold” side of our current climate. In any case, the existence, position and properties of such tipping points need to be established by physically careful and mathematically rigorous studies. The “margins of maneuver” seem reasonably wide, at least on the time scale of tens to hundreds of years, but this does not eliminate the possibility to eventually reach one such tipping point, and thus we are led directly to the next question.

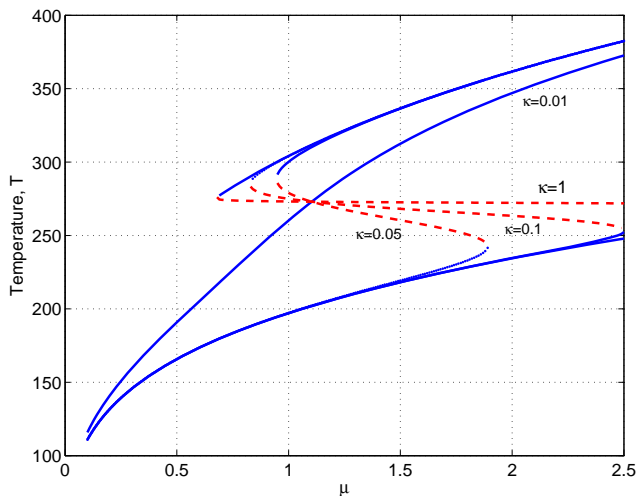


Fig. 6. Equilibrium solutions of the EBM (15) for ice-albedo feedback functions $\alpha = \alpha(T; \kappa)$ given by Eq. (13) and corresponding to the steepness parameter values $\kappa = 0.01, 0.05, 0.1$ and 1. Notice the deformation of the stable (blue, solid lines) and unstable (red, dashed lines) solution branches: the “cold” tipping point moves to the left, away from $\mu = 1.0$, as κ increases; compare Fig. 3a.

4.3 How close are we to a cold tipping point?

Let us assume for the moment that the dangers of further warming will lead humanity to actually stop, and possibly reverse, the current trend of an increasingly positive net radiation balance. Given, on the other hand, the dangers of a snowball Earth, one might want to estimate then the closeness of the climate system to the top-left bifurcation point in Fig. 5 here.

The GCM simulations of Wetherald and Manabe (1975) (see again their Fig. 5) suggest that this point might lie no farther than 5% below the current value of the solar constant. At the same time, the Sun has been much fainter 4 Gyr ago (by approximately 25–30%) than today, without the Earth ending up in a deep freeze, except possibly much later. So how close are we to this tipping point?

Figure 6 here shows stable and unstable equilibrium solutions for different profiles of the ice-albedo feedback, $\alpha = \alpha(T; \kappa)$; this profile is determined by the value of the steepness parameter κ (cf. Fig. 3a). The figure suggests that the steeper the ramp of the ice-albedo feedback function, i.e. the larger κ , the further away the bifurcation might lie. In fact, for a very smooth dependence of the albedo on temperature, i.e. for a very small κ , there is no bifurcation at all (not shown): very small values of κ produce only a single-valued, smoothly increasing, stable equilibrium solution to Eq. (15) for any value of μ .

It seems worthwhile to carry out systematic bifurcation studies with atmospheric, oceanic and coupled GCMs to examine this question more carefully, for warm tipping points, as well as for cold ones. Such studies are made possible by

current computing capabilities, along with well-developed methods of numerical bifurcation theory (Dijkstra and Ghil, 2005; Simonnet et al., 2009). This approach holds some promise in evaluating the distance of the current climate state from either a catastrophic warming or a catastrophic cooling.

Acknowledgements. We thank two anonymous referees for their constructive comments that helped improve the paper. This study was supported by US Department of Energy grants DE-FG02-07ER64439 and DE-FG02-07ER64440 from its Climate Modeling Programs.

Edited by: H. A. Dijkstra

Reviewed by: P. Ditlevsen and another anonymous referee

References

- Allen, M. R. and Frame, D. J.: Call off the quest, *Science*, 318(5850), 582–583, 2007.
- Andronov, A. A. and Pontryagin, L. S.: *Systèmes grossiers*, Dokl. Akad. Nauk. SSSR, 14(5), 247–250, 1937.
- Arfken, G.: *Taylor’s Expansion, Mathematical Methods for Physicists*, 3rd ed., Orlando, FL, Academic Press, pp. 303–313, 1985.
- Bryan, F.: High-latitude salinity effects and interhemispheric thermohaline circulations, *Nature*, 323, 301–304, 1986.
- Budyko, M. I.: The effect of solar radiation variations on the climate of the Earth, *Tellus*, 21, 611–619, 1969.
- Cess, R. D.: Climate change: An appraisal of atmospheric feedback mechanisms employing zonal climatology. *J. Atmos. Sci.*, 33, 1831–1843, 1976.
- Cess, R. D., Potter, G. L., Blanchet, J. P., Boer, G. J., Ghan, S. J., Kiehl, J. T., Le Treut, H., Li, Z.-X., Liang, X.-Z., Mitchell, J. F. B., Morcrette, J.-J., Randall, D. A., Riches, M. R., Roeckner, E., Schlese, U., Slingo, A., Taylor, K. E., Washington, W. M., Wetherald, R. T., and Yagai, I.: Interpretation of cloud-climate feedbacks as produced by 14 atmospheric general circulation models, *Science*, 245, 513–551, 1989.
- Constantin, P., Foias, C., Nicolaenko, B., and Temam, R.: *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations*, Springer-Verlag, New York, 122 pp., 1989.
- Crafoord, C. and Källén, E.: Note on condition for existence of more than one steady-state solution in Budyko-Sellers type models, *J. Atmos. Sci.*, 35, 1123–1125, 1978.
- Crutzen, P. J.: Albedo enhancement by stratospheric sulfate injections: A contribution to resolve a policy dilemma?, *Climatic Change*, 77, 211–219, doi:10.1007/s10584-006-9101-y, 2006.
- Dijkstra, H. A. and Ghil, M.: Low-frequency variability of the large-scale ocean circulation: A dynamical systems approach, *Rev. Geophys.*, 43, RG3002, doi:10.1029/2002RG000122, 2005.
- Donnadieu, Y., Goddérès, Y., Ramstein, G., Nédélec, A., and Meert, J.: A Snowball Earth climate triggered by continental break-up through changes in runoff, *Nature*, 428, 303–306, 2004.
- Gates, W. L. and Mintz, Y. (Eds.): *Understanding Climatic Change: A Program for Action*, National Academies Press, Washington, D.C., 239 pp, 1975.
- Ghil, M.: Climate stability for a Sellers-type model, *J. Atmos. Sci.*, 33, 3–20, 1976.
- Ghil, M.: Cryothermodynamics: The chaotic dynamics of paleoclimate, *Physica D*, 77, 130–159, 1994.

- Ghil, M. and Childress, S.: Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory, and Climate Dynamics, Springer-Verlag, 485 pp., 1987.
- Ghil, M. and Robertson, A. W.: Solving problems with GCMs: General circulation models and their role in the climate modeling hierarchy, in: General Circulation Model Development: Past, Present and Future, edited by: Randall, D., Academic Press, San Diego, 285–325, 2000.
- Ghil, M., Chekroun, M. D., and Simonnet, E.: Climate dynamics and fluid mechanics: Natural variability and related uncertainties, *Physica D*, 237, 2111–2126, doi:10.1016/j.physd.2008.03.036, 2008.
- Gladwell, M.: The Tipping Point: How Little Things Can Make a Big Difference, Little Brown, 2000.
- Hannart, A., Dufresne, J.-L., and Naveau, P.: Why climate sensitivity may not be so unpredictable, *Geophys. Res. Lett.*, 36, L16707, doi:10.1029/2009GL039640, 2009.
- Held, I. M.: The gap between simulation and understanding in climate modeling, *B. Am. Meteorol. Soc.*, 86, 1609–1614, 2005.
- Held, I. M. and Suarez, M. J.: Simple albedo-feedback models of ice-caps. *Tellus*, 26, 613–629, 1974.
- Held, I. M. and Soden, B. J.: Water vapor and global warming, *Annu. Rev. Energy Environ.*, 25, 441–475, 2000.
- Hillerbrand, R. and Ghil, M.: Anthropogenic climate change: Scientific uncertainties and moral dilemmas, *Physica D*, 237(14–17), 2132–2138, 2008.
- Hoffman, P. F., Kaufman, A. J., Halverson, G. P., and Schrag, D. P.: A Neoproterozoic snowball earth, *Science*, 281, 1342–1346, 1998.
- Hyde, W. T., Crowley, T. J., Baum, S. K., and Peltier, W. R.: Neoproterozoic “snowball Earth” simulations with a coupled climate/ice-sheet model, *Nature*, 405, 425–429, 2000.
- Lenton, T. M., Held, H., Kriegler, E., Hall, J. W., Lucht, W., Rahmstorf, S., and Schellnhuber, H. J.: Tipping elements in the Earth’s climate system, *Proc. Natl. Acad. Sci. USA*, 105, 1786–1793, 2008.
- Lorenz, E. N.: Deterministic nonperiodic flow, *J. Atmos. Sci.*, 20, 130–141, 1963.
- MacCracken, M. C.: Geoengineering: Worthy of Cautious Evaluation?, *Clim. Change*, 77, 235–243, 2006.
- McWilliams, J. C.: Irreducible imprecision in atmospheric and oceanic simulations, *Proc. Natl. Acad. Sci. USA*, 104, 8709–8713, 2007.
- North, G. R.: Theory of energy-balance climate models, *J. Atmos. Sci.*, 32, 2033–2043, 1975.
- North, G. R., Cahalan, R. F., and Coakley, J. A.: Energy-balance climate models, *Rev. Geophys.*, 19, 91–121, 1981.
- Pierrehumbert, R. T.: High levels of atmospheric carbon dioxide necessary for the termination of global glaciation, *Nature*, 429, 646–649, 2004.
- Poulsen, C. J. and Jacob, R. L.: Factors that inhibit snowball Earth simulation, *Paleoceanography*, 19, PA4021, doi:10.1029/2004PA001056, 2004.
- Quon, C. and Ghil, M.: Multiple equilibria in thermosolutal convection due to salt-flux boundary conditions, *J. Fluid Mech.*, 245, 449–483, 1992.
- Ramanathan, V., Crutzen, P. J., Kiehl, J. T., and Rosenfeld, D.: Aerosols, climate, and the hydrological cycle, *Science*, 294(5549), 2119–2124, 2001.
- Roe, G. H. and Baker, M. B.: Why is climate sensitivity so unpredictable?, *Science*, 318(5850), 629–632, 2007.
- Ruelle, D. and Takens, F.: On the nature of turbulence, *Commun. Math. Phys.*, 20, 167–192, 1971.
- Schlesinger, M. E.: Feedback analysis of results from energy balance and radiative-convective models, in: Projecting the Climatic Effects on Increasing Carbon Dioxide, edited by: MacCracken, M. C. and Luther, F. J., DOE/ER-0237 US Department of Energy, Washington D.C., 280–319, 1985.
- Schlesinger, M. E.: Equilibrium and transient climatic warming induced by increased atmospheric CO₂, *Clim. Dynam.*, 1, 35–51, 1986.
- Schneider, S. H. and Dickinson, R. E.: Climate modeling, *Rev. Geophys. Space Phys.*, 25, 447–493, 1974.
- Sellers, W. D.: A climate model based on the energy balance of the earth-atmosphere systems, *J. Appl. Meteorol.*, 8, 392–400, 1969.
- Simonnet, E., Dijkstra, H. A., and Ghil, M.: Bifurcation analysis of ocean, atmosphere and climate models, in: Computational Methods for the Ocean and the Atmosphere, edited by: Temam, R. and Tribbia, J. J., 2009.
- Smale, S.: Differentiable dynamical systems, *B. Am. Math. Soc.*, 73, 199–206, 1967.
- Study of Man’s Impact on Climate (SMIC): Inadvertent Climate Modification, MIT Press, Cambridge (Mass.), 1971.
- Stommel, H.: Thermohaline convection with two stable regimes of flow, *Tellus*, 13, 224–230, 1961.
- Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 2nd Ed., 648 pp, 1997.
- Thual, O. and McWilliams, J. C.: The catastrophe structure of thermohaline convection in a two-dimensional fluid model and a comparison with low-order box models, *Geophys. Astrophys. Fluid Dynam.*, 64, 67–95, 1992.
- Watkins, N. W. and Freeman, M. P.: Geoscience – Natural complexity, *Science*, 320, 323–324, 2008.
- Wetherald, R. T. and Manabe, S.: The effect of changing the solar constant on the climate of a general circulation model, *J. Atmos. Sci.*, 32, 2044–2059, 1975.